

Analytical Expressions for Effective Indices of Modes of Optical Fibers Near and Beyond Cutoff

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Abstract—We derive an analytical expression for the effective indices of modes of circular step-index fibers valid near their cutoff wavelengths. The approximation, being a first-order Taylor series of a smooth function, is also valid for the real part of the effective index beyond cutoff where the modes become lossy. The approximation is used to derive certain previously unknown mode properties. For example, it is shown that for non-dispersive materials the EH-mode group index at cutoff, surprisingly, does not depend on wavelength, core radius, or even radial mode order.

Index Terms—Step-Index Fiber, Modes, Mathematical Methods

I. INTRODUCTION

CIRCULAR step-index fibers are the simplest type of optical fiber, and they consist of a circular core of high-index medium surrounded by a cladding of lower refractive index. Whereas graded-index fibers have dominated the multimode fiber scene in recent decades, step-index fibers have been shown to support optical phenomena not possible in graded-index fibers. A notable example of such a phenomenon is soliton self-mode conversion, in which a soliton spontaneously changes color and switches from one spatial mode to another [1], [2], [3], making step-index multimode fibers useful for the generation of energetic dual- or multicolor [4] ultrashort optical pulses at unconventional wavelengths, for example.

Linear light propagation in step-index fibers is determined by its color and its spatial mode content, with different spatial modes having different effective refractive indices. The study and modelling of light in multi-mode fibers thus requires determination of the effective indices of the modes. The determination of modes of circular step-index fibers can be done semi-analytically by exploiting the fact that the radial dependence of the electric and magnetic fields making up the modes are described by Bessel functions. Even still, the determination of modal parameters becomes computationally intensive and oftentimes also numerically inaccurate for heavily multimode fibers. Furthermore, modelling pulse propagation in fibers requires knowledge on the effective index for a range of wavelengths, thus increasing the number of times the effective index has to be computed. Therefore, tools to speed up the calculations can be invaluable, especially when looking for fiber designs with specific properties.

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Any mode except for the fundamental fiber mode has a cutoff wavelength, and the mode becomes lossy for wavelengths longer than the cutoff wavelength. Near cutoff, the dispersion properties of the modes can change drastically with wavelength, and higher-order modes can be used for dispersion compensation in this wavelength regime [5]. It has recently been shown [6], that for modes with high azimuthal order (orbital angular momentum, OAM) and low radial order, the losses beyond cutoff can be small enough to allow these leaky modes to be useful for pulse transmission. Furthermore, these leaky modes are much more stable against fiber imperfections, as mode mixing is mitigated due to the fact that high radial orders experience larger losses beyond cutoff, making random coupling from low radial orders to higher orders negligible.

The behavior of modes near the cutoff wavelength is therefore of interest not just because of the exotic dispersive properties but also due to the fact that the properties of high-OAM modes are more desirable beyond cutoff than those of conventional modes below the cutoff wavelength. Here, we derive a linear approximation to the effective indices of modes near their cutoff wavelengths, simply expressing the effective index as a function of wavelength and fiber parameters. The approximation can be used to speed up the determination of effective indices, remains valid beyond cutoff, and can be used to determine mode properties at cutoff analytically. As an example, we derive certain asymptotic formulae for mode behavior when their radial and/or azimuthal order is increased, as well as a curious result showing that the group index of EH-modes at cutoff is independent of waveguide dispersion, i.e., in the absence of material dispersion, it does not depend on fiber core size, the cutoff wavelength, or even radial mode order.

II. FIRST-ORDER APPROXIMATION TO THE MODAL DETERMINANT EQUATION

Let the core radius be a , core index n_1 , and cladding index n_2 . We focus on the effects of waveguide dispersion, and material dispersion is neglected by assuming that the indices n_1 and n_2 are constants. The cladding is assumed to be infinite, which is a valid approximation for a plethora of cases of interest. The validity of the approximation essentially requires that the cladding is thick enough such that the mode fields have decayed to negligible levels by the edge of the cladding. Obviously, the thicker the cladding, the better the approximation, but what constitutes as “thick enough” also depends on the mode orders: modes of high radial order and/or low azimuthal order decay slower away from the core and therefore require

physically thicker claddings. Such modes are thus more prone to become cladding-guided beyond cutoff, in which case the whole fiber essentially acts as the “core” in the mathematical sense, and the medium surrounding the cladding (generally a coating and/or air) as the “cladding”. All modes will eventually become cladding-guided for long enough wavelengths, but the infinite cladding remains a standard textbook approximation and works especially well for modes of high azimuthal order due to the rapid field decay in the radial direction outside of the core, even beyond cutoff [6].

Let us define the usual normalized frequency as $V = ak_0\sqrt{n_1^2 - n_2^2}$, where $k_0 = 2\pi/\lambda_0$ is the vacuum wavenumber. In cylindrical coordinates r, ϕ, z , the z -components of the electric and magnetic fields are of the functional form

$$E_z(r, \phi, z, t) = AJ_m(pr)e^{im\phi}e^{i(\beta z - \omega t)}, \quad r \leq a \quad (1)$$

$$H_z(r, \phi, z, t) = BJ_m(pr)e^{im\phi}e^{i(\beta z - \omega t)}, \quad r \leq a \quad (2)$$

$$E_z(r, \phi, z, t) = CK_m(qr)e^{im\phi}e^{i(\beta z - \omega t)}, \quad r \geq a \quad (3)$$

$$H_z(r, \phi, z, t) = DK_m(qr)e^{im\phi}e^{i(\beta z - \omega t)}, \quad r \geq a \quad (4)$$

where $A, B, C, D, m, \beta, \omega$ are constants, m being the azimuthal order, $\beta = n_{\text{eff}}k_0 = 2\pi n_{\text{eff}}/\lambda_0$ the propagation constant, and ω the angular frequency. J_m is the Bessel function of the first kind, and K_m is the modified Bessel function of the second kind. The functional form is dictated by Maxwell’s equations and the cylindrical symmetry. The other field components, E_ϕ, E_r, H_ϕ, H_r , can be obtained from E_z and H_z . The azimuthal components E_ϕ, E_z, H_ϕ, H_z need to be continuous across the core-cladding interface, which then ties the constants A, B, C , and D to one another. In matrix form, these continuity conditions can be written as

$$M \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (5)$$

where

$$M = \begin{bmatrix} J_m & 0 & -K_m & 0 \\ 0 & J_m & 0 & -K_m \\ i\frac{m\beta}{ap^2}J_m & -\frac{\omega\mu_1}{p}J'_m & i\frac{m\beta}{aq^2}K_m & -\frac{\omega\mu_2}{q}K'_m \\ \frac{\omega\varepsilon_1}{p}J'_m & i\frac{m\beta}{ap^2}J_m & \frac{\omega\varepsilon_2}{q}K'_m & i\frac{m\beta}{aq^2}K_m \end{bmatrix}, \quad (6)$$

where

$$p = k_0\sqrt{n_1^2 - n_{\text{eff}}^2} \quad (7)$$

$$q = k_0\sqrt{n_{\text{eff}}^2 - n_2^2}, \quad (8)$$

and $J_m = J_m(ap)$, $J'_m = J'_m(ap)$, $K_m = K_m(aq)$, and $K'_m = K'_m(aq)$. Equation (5) has a non-trivial solution if and only if $\det(M) = 0$. The propagation constants β can then be determined from this condition numerically.

All modes except for the fundamental mode have a cutoff wavelength, at which the effective index n_{eff} becomes equal to the cladding index and beyond which the mode becomes lossy. The effective index is a smooth function of the wavelength, and hence near the cutoff wavelength the effective index can be written as $n_{\text{eff}} = n_2 + d$, where d is small. This can then be plugged into the equation $\det(M) = 0$. Since d is small,

the determinant is well approximated by a series expansion in d centered around $d = 0$. The Bessel functions K_m and K'_m and the factor $1/q$ diverge at $d = 0$, so the series expansion will not be a Taylor series, but a Puiseux series. The Bessel functions J_m and J'_m have simple Taylor expansions at $d = 0$, but for K_m the following series needs to be used:

$$K_m(qa) = \frac{1}{(aq)^m} [D_m + F_m(aq)^2 + G_m(aq)^4 + \mathcal{O}(d^{3-\delta})], \quad (9)$$

where

$$D_m = 2^{m-1}(m-1)! \quad (10)$$

$$F_m = -2^{m-3}(m-2)! \quad (11)$$

$$G_m = 2^{m-6}(m-3)! \quad (12)$$

and δ is an arbitrarily small positive real number. The expansion is valid for $m \geq 3$. For $m \geq 4$ we can set $\delta = 0$, but for $m = 3$ it is needed to account for an error term proportional to $d^3 \ln d$. A similar expansion is valid for $m \leq 2$, but the factors D_m, F_m , and G_m will depend on $\ln(aq)$. The same principles can be applied to modes with $m \leq 2$, but the logarithm terms complicate the mathematics, leading to having to express d through the Lambert W -function in the end. In what follows, it will therefore be assumed $m \geq 3$. We can now plug in Eq. (9) into $\det(M) = 0$ and expand all the other factors in the determinant as a series in d as well. When keeping only the two dominant order terms in d , the modal equation can be written in the form

$$0 = (aq)^N \det(M) \approx a_{M,m} + b_{M,m}d, \quad (13)$$

where N is an integer, $a_{M,m}$ and $b_{M,m}$ are constants that depend on the wavelength and fiber parameters. The equation is trivially solved for d , yielding an approximation for n_{eff} valid near the cutoff frequency. The full derivation can be found in the supplement at <https://arxiv.org/pdf/2411.01154#page=6>, but the final result is

$$n_{\text{eff}}^{(\text{appr.})} = n_2 - \frac{mJ_m V n_2^2 J_m - (m-1)(n_1^2 + n_2^2)J_{m-1}}{n_2 S_1 J_{m+1}^2 + S_2 V J_{m+1} J_m + S_3 J_m^2} V^3, \quad (14)$$

where

$$S_1 = (m+2)(m-1)V^4 + 2f(m^2-1)V^2 \quad (15)$$

$$S_2 = -(m-1)V^4 - 2(2m^3 - m^2 - m - f)V^2 - 8fm^2(m-1) \quad (16)$$

$$S_3 = \left[m^2 - \frac{m-1}{m-2}f \right] V^4 + 4m^2(m-1)^2V^2 + 8fm^2(m-1)^2 \quad (17)$$

$$f = a^2 k_0^2 n_2^2, \quad (18)$$

and the Bessel functions J_m and J_{m-1} are evaluated at V . Note that Eq. (14) contains the cutoff conditions for HE- and EH-modes: $n_{\text{eff}}^{(\text{appr.})} = n_2$ if and only if

$$J_m(V) = 0, \quad (19)$$

which is the cutoff condition for EH-modes, or if

$$Vn_2^2 J_m - (m-1)(n_1^2 + n_2^2)J_{m-1} = 0, \quad (20)$$

which is the cutoff condition for HE-modes. Let us denote the solutions of Eqs. (19) and (20) in ascending order by j_{mn} and s_{mn} , respectively, so that j_{m1} is the smallest positive zero of J_m , for example.

III. BEHAVIOR OF MODES NEAR CUTOFF

The expression for the approximate effective index in Eq. (14) is evidently very nonlinear in the vacuum wavelength λ_0 , but, being a first-order expansion in d , gives an approximation that is locally co-linear to the actual effective index with respect to wavelength at cutoff. Equation. (14) can therefore be utilized to further derive approximations of the form

$$n_{\text{eff}} \approx n_2 + \kappa(\lambda_0 - \lambda_c), \quad (21)$$

where κ is a constant and λ_c is a cutoff wavelength. The cutoff wavelengths are related to j_{mn} and s_{mn} as

$$\lambda_c = \frac{2\pi a}{x_{mn}} \sqrt{n_1^2 - n_2^2}, \quad (22)$$

where $x_{mn} = j_{mn}$ for EH-modes and $x_{mn} = s_{mn}$ for HE-modes. Again, the derivation of the following results can be found in the supplement (<https://arxiv.org/pdf/2411.01154#page=6>), but the approximate effective indices for HE-modes can be written as

$$n_{\text{eff}}^{\text{HE}} - n_2 \approx \left(1 - \frac{\lambda_0}{\lambda_c}\right) \frac{m(n_1^4 - n_2^4)}{n_2} \quad (23)$$

$$\times \frac{(m-1)^2(n_1^4 - n_2^4) + n_2^4 s_{mn}^2}{(n_1^2 - n_2^2)P_m + n_2^2 Q_m s_{mn}^2}, \quad (24)$$

where

$$P_m = m(m-1)(m-2)n_1^2 + m^2(m-1)n_2^2 \quad (25)$$

$$Q_m = \frac{m-1}{m-2}(n_1^2 + n_2^2)^2 + (m-2)n_2^2(n_1^2 + n_2^2) + 2n_2^4. \quad (26)$$

The expression for EH-modes is simpler:

$$n_{\text{eff}}^{\text{EH}} - n_2 \approx \left(1 - \frac{\lambda_0}{\lambda_c}\right) \frac{m(n_1^4 - n_2^4)}{n_2[(m+2)n_1^2 + mn_2^2]}. \quad (27)$$

Equations (24) and (27) are linear approximations to the effective index at the cutoff wavelength. They can therefore be used to analytically determine the *exact* group index n_g at any cutoff wavelength through

$$n_g(\lambda_c) = n_2 - \lambda_c \left[\frac{dn(\lambda_0)}{d\lambda_0} \right]_{\lambda_0=\lambda_c}. \quad (28)$$

This yields

$$n_g^{\text{HE}} = n_2 + \frac{m(n_1^4 - n_2^4)}{n_2} \frac{(m-1)^2(n_1^4 - n_2^4) + n_2^4 s_{mn}^2}{(n_1^2 - n_2^2)P_m + n_2^2 Q_m s_{mn}^2}, \quad (29)$$

and

$$n_g^{\text{EH}} = n_2 + \frac{m(n_1^4 - n_2^4)}{n_2[(m+2)n_1^2 + mn_2^2]}. \quad (30)$$

A surprising result can be seen from Eq. (30): *The group velocity of an EH mode at the cutoff wavelength does not*

depend on the cutoff wavelength, core radius, or radial mode order. As a reminder, material dispersion has been neglected in the derivation, and the result of Eq. (30) essentially means that the EH mode group velocity at cutoff is unaffected by waveguide dispersion. Since Eqs. (24) and (27) are first-order approximations to the effective index, the inclusion of material dispersion would simply add terms proportional to the wavelength slopes $dn_1/d\lambda$ and $dn_2/d\lambda$ to them.

Equations (24) and (27) can also be used to determine mode behavior near cutoff when the azimuthal and/or radial mode order is increased. The azimuthal mode order is arguable the more interesting one, since it has been demonstrated that modes of high azimuthal order can propagate with small loss beyond cutoff. Equations (24) and (27) work beyond the cutoff wavelength as well, since they are simply linear approximations to a smooth function, and they give the approximate real part of the effective index on both sides of the cutoff wavelength. In the mathematical sense, modes exist beyond cutoff, but they would require infinite energy, which is why they become lossy in real life.

Consider, first, the simpler EH-modes for large azimuthal order m and of radial order one. The cutoff wavelengths of these modes are related to J_{m1} , and j_{m1} obey inequalities given by [7]

$$m + km^{1/3} + \frac{0.5}{m^{1/3}} \geq j_{m1} \geq m + km^{1/3} + \frac{1.357}{m^{1/3}} \quad (31)$$

for $m \geq 1$, where $k = 1.855757\dots$. Plugging these into Eq. (27) then gives upper and lower bounds for the effective indices near the cutoff. The upper bound is particularly useful when determining the effective indices numerically, as below the cutoff wavelength the effective index is naturally bound by n_2 from below. The inequalities for j_{mn} also show that

$$n_{\text{eff}}^{\text{EH}} \rightarrow \frac{n_2^2}{n_2} - \lambda_0 \frac{m + 1.855757m^{1/3}}{2\pi a n_2} \sqrt{n_1^2 - n_2^2} \quad (32)$$

as $m \rightarrow \infty$. This gives the asymptotic behavior of EH-modes near cutoff for large orbital angular momenta.

As for the HE modes, first note that the cutoff condition can be written as

$$\frac{n_2^2 + n_2^2}{n_1^2 - n_2^2} J_{m-2}(V_c) = J_m(V_c). \quad (33)$$

The zeros of J_{m-2} and J_m coalesce (i.e. $j_{(m-2)n} - j_{m(n-1)} \rightarrow 0$) as $n \rightarrow \infty$, which means $s_{mn} \rightarrow j_{(m-2)n}$ as $n \rightarrow \infty$. Furthermore, for most fibers, the factor in front of $J_{m-2}(V_c)$ is very large, which means that the solutions s_{mn} are close to $j_{(m-2)n}$ even for small radial orders n . We can therefore make the approximation

$$s_{m1} \approx m - 2 + 1.855757(m-2)^{1/3}. \quad (34)$$

Plugging this approximation into Eq. (24) gives

$$n_{\text{eff}}^{\text{HE}} - n_2 \approx \left(1 - \frac{m-2 + 1.855757(m-2)^{1/3}}{2\pi a \sqrt{n_1^2 - n_2^2}} \lambda_0\right) \times \frac{n_1^4}{n_2} \frac{n_1^4 - n_2^4}{n_1^4 + n_1^2 n_2^4 + n_2^6 - n_2^4}, \quad (35)$$

as $m \rightarrow \infty$. The weak guidance approximation ($n_1 \approx n_2$) could be dropped with a better approximation for s_{mn} in terms of m , n_1 , and n_2 .

Zeros of J_m far on the positive real axis also have an asymptotic approximation [8]:

$$j_{mn} = \left(n + \frac{m}{2} - \frac{1}{4} \right) \pi - \frac{4m^2 - 1}{(8n + 4m - 2)\pi} + \mathcal{O}(n^{-3}) \quad (36)$$

that can be plugged into Eqs. (24) and (27) to see how the behavior of effective index near cutoff changes with radial order. This approximation is also useful in determining the cutoff wavelengths themselves, as it provides a good starting guess for any numerical root-finding algorithm. Equation (36) shows that for large radial orders, the wavelength-slope of the effective index increases slightly faster than but approximately linearly with radial mode order.

IV. DISCUSSION

As an example, consider a fiber with core radius $a = 20 \mu\text{m}$, core index $n_1 = 1.45$ and cladding index 1.44. Figure 1 shows the effective indices of the supported HE modes of azimuthal order $m = 10$. The lowest radial order is 1 and the corresponding mode has the highest effective index.

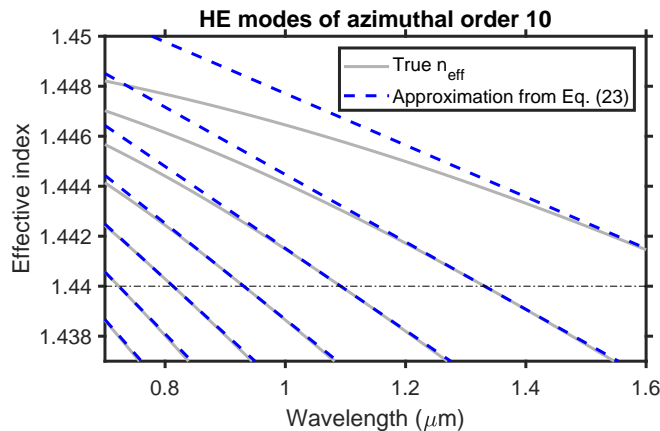


Fig. 1. The effective index of HE-modes of azimuthal order $m = 10$ in the example fiber. The radial mode order starts at one at the top (highest effective index) and increases from top to bottom. The solid gray lines show the actual effective indices, and the dashed blue lines are the linear approximations from Eq. (24). The horizontal dot-dash line indicates the cladding index, below which the modes are lossy.

The linear approximation works very well near the cutoff wavelengths, as can be seen in the figure. The same is true for EH-modes utilising Eq. (27). This case is shown in Fig. (2).

To see how well the approximation in Eq. (36) works with that of Eq. (27), consider EH-modes of azimuthal order $m = 5$ in the same example fiber. The lower azimuthal order allows for more radial orders to be supported in the fiber. Figure 3 shows the true effective indices and approximation of Eq. (27) with and without the approximation of Eq. (36). The cascaded approximation quickly becomes indistinguishable from the true linear approximation as the radial mode order increases. This means that Eq. (27) together with Eq. (36) yield a very reasonable approximation for the effective indices of EH modes or large radial order without even having to determine

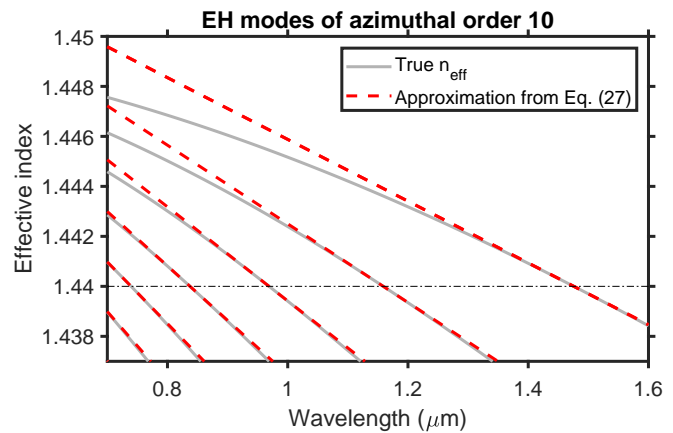


Fig. 2. The effective index of EH-modes with $m = 10$ in the example fiber. The solid gray lines show the true effective indices. The dashed red lines show the approximate effective index using Eq. (24).

or know the Bessel function zeros (mode cutoff wavelengths). Remarkably, one only needs to plug in the fiber parameters and azimuthal and radial mode order to use the approximation.

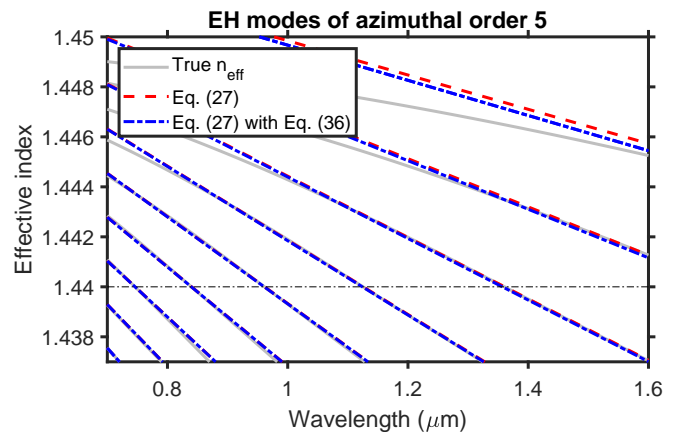


Fig. 3. The effective index of EH-modes with $m = 5$ in the example fiber. The solid gray lines show the true effective indices. The dashed red lines show the approximate effective index using Eq. (24), and the dot-dash blue line shows the cascaded approximation with Eq. (36) plugged into Eq. (24). Note that even the cascaded approximation works so well that the red dashed lines are not even visible for higher radial mode orders.

V. CONCLUSIONS, IMPLICATIONS, AND GENERALISATIONS

Linear approximations for mode effective indices near the cutoff wavelengths were derived through Taylor and Puiseux series of Bessel functions. The approximations are presented in Eqs. (24) and (27), and they have profound implications:

- 1) They are simple. They avoid the usual trial-and-error search for the effective indices, and simply give the effective index as a function of wavelength, fiber parameters, and azimuthal mode order.
- 2) They allowed for the derivation of certain surprising analytical results for the first time, such as the EH-mode group velocity at cutoff only depending on wavelength

through material dispersion, and otherwise being independent of (the cutoff) wavelength, fiber core radius, or even radial mode order.

- 3) They were used to derive previously unknown asymptotic properties of modes near cutoff as their azimuthal and/or radial mode order is increased.
- 4) They can be utilized in speeding up effective index calculations, as they provide an educated guess for the effective index in the vicinity of the cutoff wavelength, and elsewhere through extrapolation.
- 5) They remain valid beyond cutoff, where the effective indices cannot even be determined in the usual manner, and they give the real part of the effective index of lossy modes.
- 6) They exposed (and avoid) the numerical problems with determining the effective indices near cutoff the traditional way by matching the tangential fields, which results in having to subtract two very large numbers from another to yield something very small.

The method derived here also allows for some obvious generalisations. The series expansion for the determinant in the modal determinant equation was terminated after the first two dominant terms, yielding the linear approximation. Naturally, more terms could be kept in the series expansion, making it possible to analytically derive expressions for the group velocity dispersion in the vicinity of cutoff wavelengths from the quadratic approximation, for example. Furthermore, even though this manuscript only considers a simple step-index fiber consisting of a circular core and infinite cladding, the same technique can be applied to derive analytical results for more complicated fiber designs, as long as the fiber is circularly symmetric, the effective index is piecewise constant, and the fiber has cutoff conditions related to the zeros of a function (such as the Bessel function J_m here).

The circularity requirement excludes, for example, hexagonal photonic crystal fibers, but the circularity is not a strict requirement in the sense that the same techniques and ideas could be applied to rectangular waveguides and others. The piecewise-constant- n -condition excludes graded-index fibers, and the cutoff requirement excludes Bragg fibers where the guiding mechanism is not total (internal) reflection. The method is, however, readily generalizable to ring-core fibers, dispersion-compensating fibers, metal-clad hollow-core fibers, etc. Determining the modal effective indices for all such fibers always leads to the determinant of a matrix involving Bessel functions J_m , Y_m , and K_m having to equal zero, and this determinant can be expanded as a series like was done here. As a final note, since Bessel functions are associated with a lot of other physical phenomena, such as the vibrations of a drum head, the mathematical technique introduced here might offer insight and benefits in other branches of physics as well.

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