# Optical communication with structured photons propagating through dynamic, aberrating media

by

## Yiyu Zhou

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> Supervised by Professor Robert W. Boyd

The Institute of Optics Arts, Sciences and Engineering Edmund A. Hajim School of Engineering and Applied Sciences

> University of Rochester Rochester, New York

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# **Table of Contents**

Bi	ograp	bhical Sketch	V
Ac	know	vledgments	X
Ał	ostrac	et	xii
Co	ontrib	outors and Funding Sources	xiv
Li	List of Tables xvii		
Li	st of l	Figures x	viii
1	Bac	kground information	1
	1.1	Introduction to spatial modes	1
	1.2	Advances and challenges in optical communication with spatial	
		modes	3
2	Rea	lization of a Laguerre-Gauss mode sorter	10
	2.1	Fractional Fourier transform	12

	2.2	Interferometric radial mode sorter	19
	2.3	Results and discussions	24
2	Deal	ization of a Ucumita Causa made contan	21
3	Kea	ization of a Hermite-Gauss mode sorter	31
	3.1	Astigmatic mode conversion	33
	3.2	Experimental realization	35
	3.3	Results and discussions	37
4	Qua	ntum key distribution with high-dimensional spatial mode encod-	
	ing		44
	4.1	Construction of mutually unbiased bases	46
	4.2	High-dimensional spatial mode encoding and detection based on	
		a generic mode sorter	52
	4.3	Results and discussions	58
5	Digi	tal phase conjugation for free-space communication	63
	5.1	Concept of phase conjugation	67
	5.2	Realization of digital phase conjugation	68
	5.3	Results and discussions	75
6	Vect	orial phase conjugation for fiber-optic communication	85
-			0.5
	6.1	Concept of vectorial phase conjugation	85
	6.2	Implementation of vectorial phase conjugation	88
	6.3	Characterization of spatial mode fidelity	94
		6.3.1 Crosstalk matrix measurement	94

Bi	bliogr	raphy		114
7	Con	clusion	and future work	112
	6.5	Summa	ary	111
	6.4	Propos	sed spatial-mode-multiplexed communication protocol	108
		6.3.5	Stability test for vectorial phase conjugation	106
		6.3.4	Polarization crosstalk matrix	105
		6.3.3	Limitations of scalar phase conjugation	104
		6.3.2	Experimental generation fidelity of SLM	101

## **Biographical Sketch**

Yiyu Zhou was born in Hanchuan, Hubei, China. He attended School of Optical and Electronic Information at Huazhong University of Science and Technology in 2011 and graduated with a Bachelor of Engineering degree in Optoelectronic Information Engineering in 2015. He began doctoral studies in the Institute of Optics at the University of Rochester and then joined Professor Robert W. Boyd's research group, where he conducted research on mode-division multiplexing, nonlinear optics, and super-resolution imaging.

The following publications were a result of work conducted during doctoral study:

- Y. Zhou, J. Zhao, B. Braverman, K. Pang, R. Zhang, A. E. Willner, Z. Shi, and R. W. Boyd, Multiprobe time reversal for high-fidelity vortex-modedivision multiplexing over a turbulent free-space link, *Phys. Rev. Appl.*, 15, 034011, (2021).
- [2] Y. Zhou, B. Braverman, A. Fyffe, R. Zhang, J. Zhao, A. E. Willner, Z. Shi, and R. W. Boyd, High-fidelity spatial mode transmission through multimode fiber via vectorial time reversal, *arXiv: 2003.09883*, accepted by

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## Abstract

The spatial modes of photons provide a degree of freedom for boosting the channel capacity of both quantum and classical communication. Although high-speed communication using spatial modes has been reported in proof-of-principle demonstrations, the application of spatial modes in real-world communication links remains challenging due to the low detection efficiency and the high modal crosstalk induced by aberrating media. In this thesis, we discuss how to build an efficient spatial mode sorter as well as how to use phase conjugation to overcome the modal crosstalk induced by aberrating media. We begin the thesis with chapter 1 introducing the background information for spatial modes. We also review recent advances and challenges in optical communication using spatial modes. In chapter 2 we propose and demonstrate a Laguerre-Gauss (LG) mode sorter based on the fractional Fourier transform (FRFT) to efficiently decompose the optical field according to its radial mode index. In chapter 3 we propose and present the realization of an efficient, robust mode sorter that can sort a large number of Hermite-Gauss (HG) modes based on the relation between HG modes and LG modes.

Based on the spatial mode sorter we develop, in chapter 4 we develop and element a generic mode sorter that is capable of sorting the superposition of

implement a generic mode sorter that is capable of sorting the superposition of LG modes through the use of a mode converter. As a consequence, we demonstrate an 8-dimensional quantum key distribution (QKD) experiment involving all three transverse degrees of freedom: spin, azimuthal, and radial quantum numbers of photons. In chapter 5 we show that digital phase conjugation is an effective method for mitigating atmospheric turbulence. We experimentally characterize seven orbital angular momentum (OAM) modes after propagation through a 340m outdoor free-space link and observe a suppression of average modal crosstalk from 37.0% to 13.2% by implementing real-time digital phase conjugation. We implement a classical mode-division multiplexing (MDM) system as a proof-ofprinciple demonstration, and the bit error rate is reduced from  $3.6 \times 10^{-3}$  to be less than  $1.3 \times 10^{-7}$  through the use of phase conjugation. We also propose a practical and scalable scheme for high-speed, spatial-mode-multiplexed QKD through a turbulent link. In chapter 6 we show that high mode fidelity can also be achieved for a large number of spatial modes propagating through a 1-km-long, standard, graded-index, multimode fiber by using vectorial phase conjugation. Through the use of vectorial phase conjugation, we show an average mode fidelity above 80% for 210 modes over a fiber without thermal or mechanical stabilization, allowing for a channel capacity of up to 13.8 bits per sifted photon for high-dimensional QKD. We also propose a scalable spatial-mode-multiplexed QKD protocol that cannot be achieved by alternative methods. In chapter 7 we summarize the thesis and discuss the potential future work.

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Chapter 1 provides the necessary background information for this thesis and no original research is included.

Chapter 2 focuses on the realization of a radial mode sorter and is primarily adapted from a peer-reviewed publication in Physical Review Letters [1]. The work in [1] was led by Yiyu Zhou and the experiment was performed in Professor Robert W. Boyd's laboratory at the University of Rochester with contributions from Mohammad Mirhosseini, Dongzhi Fu, Jiapeng Zhao, Seyed Mohammad Hashemi Rafsanjani, Alan E. Willner, and Robert W. Boyd. This work was supported by the U.S. Office of Naval Research.

Chapter 3 focuses on the realization of a HG mode sorter and is primarily adapted from a peer-reviewed publication in Optics Letters [2]. The work in [2] was led by Yiyu Zhou and the experiment was performed in Professor Robert W. Boyd's laboratory at the University of Rochester with contributions from Jiapeng Zhao, Zhimin Shi, Seyed Mohammad Hashemi Rafsanjani, Mohammad Mirhosseini, Ziyi Zhu, Alan E. Willner, and Robert W. Boyd. This work was supported by the U.S. Office of Naval Research.

Chapter 4 focuses on the implementation of a high-dimensional QKD system and is primarily adapted from a peer-reviewed publication in Optics Express [3]. The work in [3] was led by Yiyu Zhou and the experiment was performed in Professor Robert W. Boyd's laboratory at the University of Rochester with contributions from Mohammad Mirhosseini, Stone Oliver, Jiapeng Zhao, Seyed Mohammad Hashemi Rafsanjani, Martin P. J. Lavery, Alan E. Willner, and Robert W. Boyd. This work was supported by the U.S. Office of Naval Research.

Chapter 5 focuses on the realization of MDM through a 340-m turbulent freespace link via phase conjugation and is primarily adapted from a peer-reviewed publication in Physical Review Applied [4]. The work in [4] was led by Yiyu Zhou and the experiment was performed in Professor Robert W. Boyd's laboratory at the University of Rochester with contributions from Jiapeng Zhao, Boris Braverman, Kai Pang, Runzhou Zhang, Alan E. Willner, Zhimin Shi, and Robert W. Boyd. This work was supported by the U.S. Office of Naval Research.

Chapter 6 focuses on the realization of high-fidelity transmission of spatial modes through a 1-km-long multimode fiber via vectorial phase conjugation and is primarily adapted from a work that is under review [5]. The work in [5] was led by Yiyu Zhou and the experiment was performed in Professor Robert W. Boyd's laboratory at the University of Rochester with contributions from Boris Braverman, Alexander Fyffe, Runzhou Zhang, Jiapeng Zhao, Alan E. Willner, Zhimin Shi, and Robert W. Boyd. This work was supported by the U.S. Office of Naval Research.

Chapter 7 provides the summary of this thesis and no original research is included.

# **List of Tables**

5.1 Review of previous works for free-space OAM communication .	. (	65
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6.1 Definition of the single mode index for HG modes and LG modes 86

# **List of Figures**

1.1	Intensity profile of HG and LG modes	2
2.1	Fractional Fourier transform module for mode sorting	15
2.2	Intensity profile of the generated radial modes	20
2.3	Schematic of a radial mode sorter	23
2.4	Experimental characterization of radial mode sorter	26
2.5	Crosstalk matrix of the radial mode sorter	27
3.1	Conversion relation of HG and LG modes	34
3.2	Schematic of HG mode sorter	36
3.3	Experimental results of the HG mode sorter	38
3.4	Schematic for the scalable HG mode sorter	41
4.1	Intensity profiles of spatial modes in mutually unbiased bases	47
4.2	Schematic of the generic radial superposition mode sorter	50
4.3	Experimental realization of the common-path radial mode sorter	53
4.4	Experimental realization of the radial mode converter	54
4.5	Design and characterization of the radial mode converter	55

4.6	Characterization of the mode sorter	57
4.7	Schematic of QKD experiment with high-dimensional spatial mode	
	encoding	58
4.8	Crosstalk matrix characterization for high-dimensional QKD	60
5.1	Conceptual illustration of phase conjugation	66
5.2	Experimental schematic of the phase conjugation for OAM com-	
	munication	68
5.3	Implementation of digital phase conjugation	70
5.4	Crosstalk matrix measurement by mode-multiplexed holograms .	72
5.5	Intensity profiles of the spatial modes after propagation through	
	turbulence	73
5.6	Crosstalk matrix after propagation through turbulence	74
5.7	Crosstalk matrix by using a mode spacing $\Delta \ell$ of 2	75
5.8	Eye diagrams for OAM communication	76
5.9	Transmittance measurement for the free-space link	77
5.10	Time-resolved BER for $\ell=2$ channel without phase conjugation .	78
5.11	Time-resolved BER for $\ell = 2$ channel with phase conjugation $\ . \ .$	79
5.12	Time-resolved BER for $\ell=3$ channel without phase conjugation .	80
5.13	Time-resolved BER for $\ell = 3$ channel with phase conjugation $\therefore$	81
5.14	Proposed free-space QKD protocol based on phase conjugation	83
6.1	Conceptual illustration of vectorial phase conjugation	87
6.2	The full experimental schematic of vectorial phase conjugation	89

6.3	Digital realization of vectorial phase conjugation
6.4	Experimentally measured speckle patterns
6.5	The unnormalized mode fidelity for LG and HG modes 94
6.6	Crosstalk distribution for LG and HG modes
6.7	Crosstalk matrix for LG modes without vectorial phase conjugation 96
6.8	Crosstalk matrix for HG modes without vectorial phase conjugation 97
6.9	Crosstalk matrix for LG modes with vectorial phase conjugation . 98
6.10	Crosstalk matrix for HG modes with vectorial phase conjugation . 99
6.11	Normalized mode fidelity for LG and HG modes
6.12	Experimental generation fidelity characterization
6.13	The mode fidelity for scalar phase conjugation
6.14	The spatial mode profiles with scalar phase conjugation 104
6.15	The normalized crosstalk matrix in the polarization subspace 105
6.16	Stability test for vectorial phase conjugation
6.17	Proposed spatial-mode-multiplexed QKD protocol based on vec-
	torial phase conjugation
7.1	Proposed high-speed spatial mode switch

## **Chapter 1**

# **Background information**

### **1.1 Introduction to spatial modes**

The concept of spatial modes of electromagnetic waves was developed for microwave cavities as early as 1897 by Lord Rayleigh [6] and then extended to laser cavities and optical waveguides, where spatial modes are also referred to as transverse modes [7]. Spatial modes are typically described by a set of orthonormal functions that are solutions of Maxwell's equations given certain boundary conditions. A few well-known examples of spatial modes include transverse electric (TE) modes, linearly polarized (LP) modes [8], Hermite-Gauss (HG) modes, and Laguerre-Gauss (LG) modes [9]. In the paraxial regime, the scalar Helmholtz equation in free space can be written as [10]

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + 2ik\frac{\partial E}{\partial z} = 0, \qquad (1.1)$$



Figure 1.1: Intensity profile of (a) HG modes and (b) LG modes.

where E(x, y, z) is the electric field,  $k = 2\pi/\lambda$ , and  $\lambda$  is the wavelength. It can be shown that HG modes are solutions to the paraxial wave equation in Cartesian coordinates, which can be expressed as [7]

$$\operatorname{HG}_{mn}(x, y, z) = \frac{w_0}{w(z)} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left(-\frac{x^2 + y^2}{w^2(z)}\right)$$
  
 
$$\cdot \exp\left(-i\frac{k(x^2 + y^2)}{2R(z)}\right) \exp(i\phi(z)) \exp(-ikz),$$
(1.2)

where  $w_0$  is the beam waist radius,  $z_R = \pi w_0^2 / \lambda$ ,  $w(z) = w_0 \sqrt{1 + (z/z_R)^2}$ ,  $H_m(\cdot)$  and  $H_n(\cdot)$  are Hermite polynomial of order m and n respectively [11],  $R(z) = z + (z_R^2/z)$ , and  $\phi(z) = (m + n + 1) \cdot \arctan(z/z_R)$ . The intensity profile of HG modes is presented in Fig. 1.1(a), and it can be seen that the HG modes are separable in Cartesian coordinates. The paraxial wave equation Eq. (1.1) can also be solved in cylindrical coordinates, and the corresponding solution becomes the LG modes as

$$LG_{p\ell}(r,\theta,z) = \frac{w_0}{w(z)} \left(\frac{\sqrt{2}r}{w(z)}\right)^{|\ell|} \exp\left(-\frac{r^2}{w^2(z)}\right) L_p^{|\ell|} \left(\frac{2r^2}{w^2(z)}\right)$$

$$\cdot \exp\left(-i\frac{k(x^2+y^2)}{2R(z)}\right) \exp(i\ell\theta) \exp(i\psi(z)) \exp(-ikz),$$
(1.3)

where  $L_p^{|\ell|}(\cdot)$  is the generalized Laguerre polynomial [12], p is the radial index,  $\ell$  is the azimuthal index, and  $\psi(z) = (2p + |\ell| + 1) \cdot \arctan(z/z_R)$ . The intensity profile of LG modes is presented in Fig. 1.1(b), and it can be seen that the LG modes are separable in cylindrical coordinates.

# **1.2** Advances and challenges in optical communication with spatial modes

It was realized in 1992 that the helical phase structure  $\exp(i\ell\theta)$  in LG modes is associated with the OAM of photons [13], which triggers a number of theoretical and experimental investigations for OAM modes [14–16]. Due to the orthogonality of OAM modes, different co-propagating modes can in principle be demultiplexed without modal crosstalk, and thus the data transfer rate of both quantum and classical communications can be enhanced by either MDM or high-dimensional spatial mode encoding [17]. Spatial mode encoding uses N spatial modes to encode each symbol and transmit one spatial mode at a time [18–20]. Therefore, each symbol can carry  $\log_2 N$  bits of information, and thus the information capacity can be increased by a factor of  $\log_2 N$  through the use of N spatial modes. Spatial mode encoding is often adopted in QKD because high-dimensional encoding allows for a higher error rate threshold [21], which makes QKD protocols more robust to noise and crosstalk. By contrast, MDM uses N spatial modes as N independent carriers to convey N different signal streams to the receiver through the same channel simultaneously [22, 23]. The channel information capacity can be increased by a factor of N by MDM, which is higher than  $\log_2 N$  of spatial mode encoding. Hence, MDM is more favorable to the classical communication community.

A multitude of approaches have been proposed and demonstrated to generate spatial modes for optical communication. A spatial light modulator (SLM) is a re-programmable and versatile tool that can be used to generate arbitrary spatial modes through the use of computer-generated hologram [24–30]. Liquid-crystal-on-silicon (LCoS) device [31] is a commonly used type of SLM that features phase-only modulation and high diffraction efficiency. Due to the slow response of liquid crystal molecules, LCoS device typically has a refresh rate from 60 Hz to sub kHz [32]. The LCoS SLM can also be placed inside a laser cavity to enable efficient, high-power generation of spatial modes [33]. Digital micromirror device (DMD) is another type of SLM that allows fast binary amplitude modulation. By using binary computer-generated hologram, DMD can also be used to generate arbitrary complex spatial modes [34]. The major advantage of DMD compared to LCoS device is that DMD has a refresh rate up to 20 kHz [35]. Many alternative methods and devices have also been investigated to generate spatial modes, such as q-plates [36], metasurfaces [37], photonic lanterns [38], and micro-ring

resonators [39]. However, it should noted that these methods are typically nonreconfigurable and can only generate pre-determined spatial modes.

The detection of spatial modes can be regarded as an inverse process of mode generation. Therefore, a SLM can be used to detect a spatial mode by displaying a conjugate hologram, which is referred to as projective measurement [40-42]. More specifically, for an incoming OAM mode of  $\exp(i\ell_1\theta)$ , if the SLM displays a hologram that impresses a phase modulation of  $\exp(-i\ell_2\theta)$  to the field, then the output OAM mode becomes  $\exp(i(\ell_1 - \ell_2)\theta)$ . If  $\ell_1 = \ell_2$ , then the helical phase structure can be erased and thus the beam can be coupled into a single-mode fiber (SMF) [22]. Despite the simplicity, the detection efficiency of projective measurement is intrinsically bounded to 1/N for a N-dimensional Hilbert space. By contrast, it is possible to perform projection to multiple spatial modes losslessly and simultaneously, and such a device is referred to as a mode sorter. Take the polarization degree of freedom as an example, a polarizer performs projective measurement by passing photons of a particular polarization state while absorbing photons of the other orthogonal polarization state. By contrast, a polarizing beamsplitter (PBS) can sort photons of different polarization states to different output ports losslessly. The spatial mode sorter can be realized by the optical geometrical transformation [43–45], integrated photonic waveguides [46], and the multi-plane light conversion (MPLC) [47]. Although a scalable, low-crosstalk mode sorter has been realized for OAM modes [43], sorting the full-field HG modes and LG modes with low crosstalk and high scalability remains a challenging task. It has been reported that the MPLC can be used to sort up to 210 LG modes [48], but the average modal crosstalk remains as high as 19.1%, which can be a concern particularly to quantum communication.

Despite the advances achieved in mode generation and detection, the modal crosstalk inevitably induced by transmission through aberrating media has impeded the widespread adoption of classical and quantum communication with spatial modes in practical applications. For free-space optical communication, atmospheric turbulence inevitably leads to strong modal crosstalk between spatial modes [49], which severely degrades the channel capacity of a free-space link. We next summarize several previous works to show the typical level of crosstalk between OAM modes in a turbulent, outdoor free-space channel. In a 150-m link [50], the crosstalk fluctuates between 60% and 80% depending on time. In a 300m intra-city link [20], the crosstalk is 11% with a mode spacing  $\Delta \ell$  of 4. In a 340-m cross-campus link [51], the crosstalk is measured to be in the range between 70% and 80%. In a 3-km link [52], a camera is used to measure the images of bright OAM superposition modes, and an artificial neural network (ANN) is applied for image recognition, resulting in a bit error rate of  $1.7 \times 10^{-2}$ . Hence, the turbulence can be a serious concern for crosstalk-sensitive applications such as QKD.

Adaptive optics is the most common method for turbulence correction and has been widely adopted for astronomical imaging [53]. A conventional adaptive optics system consists of a wavefront sensor and a deformable mirror at the receiver. The wavefront sensor measures the aberrated phase of an incoming beacon beam (typically a Gaussian beam), and subsequently the deformable mirror corrects the phase aberration of the incoming beam based on the feedback from wavefront sensor as post-turbulence compensation [54]. However, different OAM modes exhibit mode-dependent amplitude and phase distortions after propagation through the same turbulent link [51]. Therefore, a conventional post-turbulence single-plane phase-only adaptive optics system is unable to correct both amplitude and phase distortions for different OAM modes simultaneously, even in principle [55, 56]. Furthermore, the effectiveness of adaptive optics for OAM communication has mostly been tested in numerical simulations [56–58] or in lab-scale links with emulated, slowly varying, fully controllable turbulence [59–62]. To the best of our knowledge, there is only one experimental demonstration using adaptive optics for OAM communication through an outdoor link [51], and the crosstalk is reduced from 80% to 77% by using both adaptive optics and a fast steering mirror simultaneously. This level of crosstalk is too large to guarantee secure QKD [21]. Therefore, adaptive optics has achieved very limited performance enhancement in outdoor free-space OAM communication links despite numerous simulations and lab-scale experiments. Other methods for turbulence suppression, such as a multiple-input multiple-output (MIMO) algorithm [63] and an ANN [52], cannot be applied to QKD because these algorithms require a large number of photons for digital signal processing and thus are inappropriate for quantum applications that operate at a single-photon level.

Spatial modes have also been applied to enhance the channel capacity of fiber communication. The large number of modes supported by a standard multimode fiber (MMF) have long been recognized as an additional resource to further increase the communication rate [64–68]. It is compatible with other multiplexing methods such as wavelength-division multiplexing and can be also used to enhance quantum teleportation [69] and entanglement distribution [70, 71]. However, the inevitable mode crosstalk in standard MMFs is a persistent obstacle to practical applications of spatial modes for QKD. Tremendous efforts have been devoted to attempts to mitigate the effects of spatial mode crosstalk during the past decades. Transfer matrix inversion is a standard method that has been successfully used to transmit spatial modes through MMFs [72–78]. However, standard MMFs can support between tens and hundreds of modes depending on the wavelength, and thus the number of complex-valued elements in the transfer matrix is typically between  $10^3$  and  $10^5$ . As a consequence, all transfer matrix inversion experiments reported in the literature have used a short MMF ( $\approx 1$  m) [72–78] because the fiber has to be carefully stabilized during the slow characterization process. When applying this method to a long fiber, it is foreseeable that instability will severely impede long-distance communication outside the laboratory. By contrast, modegroup excitation [79–81] has been applied to long fibers due to the relatively low inter-modal-group crosstalk. However, for a fiber supporting N spatial modes, only approximately  $\sqrt{N}$  mode groups are supported. Thus the number of usable mode groups is intrinsically limited in this method. MIMO algorithm is another standard method for fiber crosstalk mitigation [82]. However, it require a high signal-to-noise ratio for digital signal processing and thus is unsuitable for quantum applications.

In brief, while many advances have been achieved, the detection and transmis-

sion of spatial modes remain challenging for practical applications. To address these challenges, we describe how to build an efficient mode sorter for radial LG modes in chapter 2 through the use of fractional Fourier transform. In chapter 3 we demonstrate how to sort HG modes by utilizing the conversion between HG modes and LG modes through a pair of cylindrical lenses. In chapter 4 we show how to implement QKD using high-dimensional encoding involving the entire transverse degrees of freedom of light. we discuss digital phase conjugation in chapter 5 and show that modal crosstalk induced by atmospheric turbulence can be effectively suppressed by phase conjugation. In chapter 6 we develop the concept of vectorial phase conjugation, which is an essential tool to enable high-fidelity spatial mode transmission through a 1-km-long MMF.

### Chapter 2

# **Realization of a Laguerre-Gauss mode sorter**

In the recent years, the transverse structure of optical photons has been established as a resource for storing and communicating quantum information [83]. In contrast to the two-dimensional Hilbert space of polarization, it takes an unbounded Hilbert space to provide a mathematical representation for the transverse structure of the optical field. The large information capacity of structured photons has been recently utilized to enhance QKD [84–86] and a multitude of other applications [87–89]. The OAM modes have become increasingly popular for implementing multi-dimensional quantum states due to the relative ease in generation [18], manipulation [90], and characterization of these modes [91, 92].

Although the OAM modes provide a basis set for representing the azimuthal structure of photons, they cannot completely span the entire transverse state space, which encompasses an extra (radial) degree of freedom. The LG mode functions provide a basis to fully represent the spatial structure of the transverse field [84, 93]. These modes are characterized by two numbers, the radial mode index  $p \in \{0, 1, 2, ...\}$  and the azimuthal mode index  $\ell \in \{0, \pm 1, \pm 2, ...\}$ . While the az-

imuthal number  $\ell$  is well studied due to its association with the OAM of light [13], the radial index p has so far remained relatively unexplored. In particular, it has been shown that OAM modes only constitute a subspace of LG modes and cannot be used to reach the channel capacity limit of a communication link [94]. The quantum coherence of photons in a superposition of orthogonal radial modes has been recently demonstrated in the context of quantum communication and high-dimensional entanglement [84, 89, 95]. The radial LG modes also hold a number of promising features, and have been studied in the contexts of self-healing [96], optical tweezer trapping [97], super-resolution [98], and hyperbolic momentum charge [99]. Despite the growing theoretical interests in utilizing the radial structure of photons, the experimental realizations have thus far been impeded because of the difficulty of measuring these modes.

The initial step in characterization of the radial degree of freedom of light is to find a radial mode spectrum, i.e. to find the probability P(p) of having the state prepared in mode index p. This information can be, in principle, obtained by performing a series of projective measurements. However, the most straightforward method for implementing the projective measurement of a radial LG mode requires shaping the amplitude of the incoming light beam, and the resulting loss makes this approach unsuitable for operation at the single-photon level [41]. In addition to this technical difficulty, the projective measurement of a photon results in its absorption [89]. This inherently limits the success rate to 1/d in a ddimensional state space, a rate that does not scale well with the size of the Hilbert space. An alternative approach for characterizing the radial mode structure is to sort an unknown incoming photon by its radial quantum number. A radial mode sorter would route the photon to a distinct output that is indexed by the value of its radial quantum number p, and is thus capable of performing parallel projective measurements with a success rate of unity.

#### 2.1 Fractional Fourier transform

Here, we propose and demonstrate a unitary mode sorter for the radial quantum number p. Our approach relies on a key property of the LG modes: the dependence of the effective phase velocity on the radial quantum number p. We use a set of refractive optical elements to induce the fractional Gouy phase by realizing a fractional Fourier transform (FRFT) module [100]. The FRFT module is then combined with a Mach-Zehnder interferometer that can discriminate the modes based on the magnitude of the induced phase. Our experiment can be understood as an implementation of the theoretical recipe recently developed in [101]. We provide experimental results demonstrating the ability to sort individual and superposition states residing in the 4-dimensional state space of  $p \in \{0, 1, 2, 3\}$ . Furthermore, we show that our implementation can be combined with the existing methods of sorting OAM to provide full characterization of the transverse structure of the light field.

To understand the specifics of our implementation, we examine sorting from an operational point of view. Sorting is a unitary operation that bijectively maps input photons of different modes onto different output modes. One approach to realize such an operation is by successive application of a discrete Fourier transform (i.e., F-gate), a mode-dependent phase unit (i.e.  $Z_d$ -gate), and an inverse discrete Fourier transform element [101] (Note that we use the quantum gates and the bracket notation in order to provide a concise mathematical description for the evolution of spatial modes, and not for the purpose of describing the quantum state of the electromagnetic field). The discrete Fourier transform can be realized by a combination of beam-splitters and constant-phase elements (wave plates) [102, 103]. The remaining unit required for sorting the LG modes according to their radial index is a mode-dependent phase element i.e. a  $Z_d$ -gate.

We next describe how the  $Z_d$ -gates for the LG modes can be realized using a natural property of these modes in propagation. The mathematical form of the LG modes in cylindrical coordinates at the plane of the beam waist is given by Eq. (1.3). Here we show that these modes are eigenmodes of the fractional Fourier transforms (FRFTs) [1]. The characteristic equation can be attained by using the relation between the LG function and the HG function. It is well known that the LG function can be decomposed to the HG function as [104]

$$LG_{p\ell}(r,\theta) = \sum_{k=0}^{N=m+n} i^k b_{n,m,k} HG_{N-k,k}(x,y)$$
(2.1)

where  $b_{n,m,k}$  is a constant determined by n, m and k, and its definition is

$$b_{n,m,k} = \sqrt{\frac{(N-k)!k!}{2^N n!m!}} \frac{1}{k!} \frac{d^k}{dt^k} \left[ (1-t)^n (1+t)^m \right]_{t=0}$$
(2.2)

where the relation between indices m, n and p,  $\ell$  is given by  $p = \min(m, n), \ell = m - n$  and  $2p + |\ell| = m + n$ . Since the HG function is the eigenfunction of the FRFT [105]

$$\mathcal{F}^{a}[\mathrm{HG}_{mn}(x_{0}, y_{0})] = \exp[-i(m+n)a]\mathrm{HG}_{mn}(x, y)$$
(2.3)

where a denotes the order of the FRFT and for a normal Fourier transform we have  $a = \pi/2$ . It is worth noting that a can be tuned in an experiment by using appropriate lenses. We next use Eq. (2.1) to decompose the LG function to the HG function, and then transform the resulting HG functions through Eq. (2.3). Then we can derive the transformed LG function, which can be expressed as

$$\mathcal{F}^{a}[\mathrm{LG}_{p\ell}(r_{0},\theta_{0})] = \mathcal{F}^{a}\left[\sum_{k=0}^{N=m+n} i^{k}b_{n,m,k}\mathrm{HG}_{N-k,k}(x_{0},y_{0})\right] = \sum_{k=0}^{N=m+n} i^{k}b_{n,m,k}\mathcal{F}^{a}[\mathrm{HG}_{N-k,k}(x_{0},y_{0})]$$

$$= \sum_{k=0}^{N=m+n} i^{k}b_{n,m,k}\exp(-iNa)\cdot\mathrm{HG}_{N-k,k}(x,y)$$

$$= \exp[-i(2p+|\ell|)a]\mathrm{LG}_{p\ell}(r,\theta).$$
(2.4)

Therefore, the characteristic equation can be written as

$$\mathcal{F}^{a}[\mathrm{LG}_{p\ell}(r_{0},\theta_{0})] = \exp[-i(2p+|\ell|)a]\mathrm{LG}_{p\ell}(r,\theta)$$
(2.5)



Figure 2.1: (a). Realization of the fractional Fourier transformation (FRFT) with a single lens. The LG functions are the eigenmodes of the FRFT and thus maintain their shape under this transformation. Here a p = 2 mode is shown as an example. (b). A *d*-dimensional quantum sorter composed of discrete *F*-gates and a  $Z_d$ -gate. The  $Z_d$ -gate is implemented by the FRFT in our experiment.

In the experiment we use spherical lenses and assume a symmetric two-dimensional FRFT, which is equivalent to the fractional Hankel transform [106]. The phase term here can be interpreted as a modification of the effective phase velocity of the structured beam, and is reminiscent of the Gouy phase in laser physics [7].

A simple operational unit of our mode sorter, consisting of a single lens accompanied with free-space propagation, can realize the FRFT (see Fig. 2.1(a)) [107]. It should be noted that only the LG beam of a specific beam waist radius can remain invariant after the FRFT. The beam waist radius should satisfy the condition  $\lambda \tilde{f}/\pi = w_0^2$ , where  $\tilde{f} = f \sin a$ , f is the lens focal length, a is the order of FRFT,  $\lambda$  is the wavelength, and  $w_0$  is the LG mode beam waist radius. Assume the optical field on the initial plane is  $u_0(x_0, y_0)$  and the field on the final plane is u(x, y). The field propagates a distance of z, goes through a lens of focal length f, and propagates another z again. By Fresnel propagation, the relation between  $u_0(x_0, y_0)$  and u(x, y) is [108]

$$u(x,y) \propto \exp\left[\frac{i\pi}{\lambda \tilde{f} \tan a} \left(x^2 + y^2\right)\right] \iint dx_0 dy_0 u_0(x_0,y_0)$$
$$\times \exp\left[\frac{i\pi}{\lambda \tilde{f} \tan a} \left(x_0^2 + y_0^2\right)\right] \exp\left[-\frac{2i\pi}{\lambda \tilde{f} \sin a} \left(xx_0 + yy_0\right)\right]$$
(2.6)

where  $f = \tilde{f}/\sin a$ ,  $z = \tilde{f}\tan(a/2)$  and we have omitted the normalization factor. By defining the variables  $X = x/\sqrt{\lambda \tilde{f}/2\pi}$ ,  $Y = y/\sqrt{\lambda \tilde{f}/2\pi}$ ,  $X_0 = x_0/\sqrt{\lambda \tilde{f}/2\pi}$  and  $Y_0 = y_0/\sqrt{\lambda \tilde{f}/2\pi}$ , we can rewrite the equation and express it as

$$u(x,y) \propto \exp\left[\frac{i}{2\tan a} \left(X^2 + Y^2\right)\right] \iint \exp\left[\frac{i\left(X_0^2 + Y_0^2\right)}{2\tan a}\right] \times u_0\left(\sqrt{\frac{\lambda\tilde{f}}{2\pi}}X_0, \sqrt{\frac{\lambda\tilde{f}}{2\pi}}Y_0\right) \exp\left[-\frac{i(XX_0 + YY_0)}{\sin a}\right] dX_0 dY_0$$

$$(2.7)$$
Comparing to definition of the two-dimensional FRFT [105]

$$\mathcal{F}^{a}[u_{0}(x_{0}, y_{0})] = \frac{1 - i \cot a}{2\pi} \iint dx_{0} dy_{0} u_{0}(x_{0}, y_{0})$$
$$\times \exp\left[i\left(\frac{x^{2} + y^{2}}{2 \tan a} - \frac{xx_{0} + yy_{0}}{\sin a} + \frac{x_{0}^{2} + y_{0}^{2}}{2 \tan a}\right)\right]$$
(2.8)

we could readily check that u(x, y) can be represented in the form of the FRFT as

$$u(x,y) \propto \mathcal{F}^{a}_{X_{0} \to X, Y_{0} \to Y} \left[ u_{0} \left( \sqrt{\frac{\lambda \tilde{f}}{2\pi}} X_{0}, \sqrt{\frac{\lambda \tilde{f}}{2\pi}} Y_{0} \right) \right]$$
(2.9)

The subscript  $X_0 \to X, Y_0 \to Y$  means that the FRFT is mapping the new variables  $X_0, Y_0$  to X and Y, respectively, not acting on the original x and y. Now assume the incident beam is a LG beam of the beam waist radius  $w_0$ . So we can express it as

$$u_0(x_0, y_0) = \mathrm{LG}_{p\ell} \left(\frac{r_0}{w_0/\sqrt{2}}, \theta\right) \propto \frac{1}{w_0} \left(\frac{r_0}{w_0/\sqrt{2}}\right)^{|\ell|} \times \exp\left[-\frac{1}{2} \left(\frac{r_0}{w_0/\sqrt{2}}\right)^2\right] L_p^{|\ell|} \left(\left(\frac{r_0}{w_0/\sqrt{2}}\right)^2\right) \exp(i\ell\theta)$$
(2.10)

Here we ignore the normalization factor. We can play the same trick to decompose

the LG function to the HG function as

$$u_{0}(x_{0}, y_{0}) = \mathrm{LG}_{p\ell} \left( \frac{r_{0}}{w_{0}/\sqrt{2}}, \theta \right)$$

$$= \sum_{k=0}^{N=m+n} i^{k} b_{n,m,k} \mathrm{HG}_{N-k,k} \left( \frac{x_{0}}{w_{0}/\sqrt{2}}, \frac{y_{0}}{w_{0}/\sqrt{2}} \right)$$

$$= \sum_{k=0}^{N=m+n} i^{k} b_{n,m,k} \mathrm{HG}_{N-k,k} \left( \frac{\sqrt{\lambda \tilde{f}/2\pi}}{w_{0}/\sqrt{2}} X_{0}, \frac{\sqrt{\lambda \tilde{f}/2\pi}}{w_{0}/\sqrt{2}} Y_{0} \right)$$

$$= \sum_{k=0}^{N=m+n} i^{k} b_{n,m,k} \mathrm{HG}_{N-k,k} \left( \frac{\sqrt{\lambda \tilde{f}/\pi}}{w_{0}} X_{0}, \frac{\sqrt{\lambda \tilde{f}/\pi}}{w_{0}} Y_{0} \right)$$
(2.11)

where  $p = \min(m, n)$  and  $\ell = m - n$ . Clearly, if we have  $\sqrt{\lambda \tilde{f} / \pi w_0^2} = 1$ , or equivalently  $\lambda \tilde{f} / \pi = w_0^2$ , then the field on the final plane can be simplified to

$$u(x,y) \propto \sum_{k=0}^{N=m+n} i^{k} b_{n,m,k} \mathcal{F}_{X_{0} \to X,Y_{0} \to Y}^{a} [\mathrm{HG}_{N-k,k}(X_{0},Y_{0})]$$

$$= \sum_{k=0}^{N=m+n} i^{k} b_{n,m,k} \exp(-iNa) \mathrm{HG}_{N-k,k}(X,Y)$$

$$= \exp(-iNa) \sum_{k=0}^{N=m+n} i^{k} b_{n,m,k} \mathrm{HG}_{N-k,k} \left(\frac{x}{w_{0}/\sqrt{2}}, \frac{y}{w_{0}/\sqrt{2}}\right)$$

$$= \exp[-i(2p+|\ell|)a] \mathrm{LG}_{p\ell} \left(\frac{r}{w_{0}/\sqrt{2}}, \theta\right)$$
(2.12)

We could readily see that the field on the final plane is a LG beam with the same beam waist radius  $w_0$  as long as the condition  $\lambda \tilde{f}/\pi = w_0^2$  is satisfied. Alternatively, the relation can be expressed as [1]:

$$z = \frac{\pi w_0^2}{\lambda} \tan \frac{a}{2}, \qquad f = \frac{\pi w_0^2}{\lambda \sin a}$$
(2.13)

Upon propagation through this unit, radial modes will pick up a fractional Gouy phase that depends on their respective indices. Note that the corresponding phase depends on both the radial index as well as the OAM value. This dependence does not present a problem as one can use a Dove prism to cancel the  $\ell$ -dependence and thereby retain only the *p*-dependent phase [109].

#### 2.2 Interferometric radial mode sorter

Having examined the two building blocks, i.e. the discrete Fourier transform (F-gate) and the  $Z_d$ -gate, we can design a radial index mode sorter. A schematic representation of the concept is provided in Fig. 2.1(b). Let us assume that  $\ell = 0$  for all LG modes discussed in this chapter and denote the LG mode by  $|p\rangle$ . To prepare the radial modes, we imprint the computer-generated hologram on an SLM [34]. The binary phase grating will generate the mode at the first diffraction order, which can be selected by using a Fourier-transforming lens. To verify the quality of generated mode, it is necessary to check the mode on not only the Fourier plane but also the image plane (which can accessed by performing a second successive Fourier transformation). The generated mode on two planes by our setup are presented in Fig. 2.2. We note that the polarization state of the light beam on the



#### Image plane



Figure 2.2: Generated radial modes in the Fourier plane and the image plane.

SLM should be aligned according to the SLM requirement in order to minimize interference with the strong zeroth-order light. We use a f = 1500 mm lens along with an iris to separate the first order diffracted beam. In order to increase the fidelity of the generated beam, we have applied an additional phase term to the computer generated hologram to correct for spherical aberration, astigmatism and the coma, which are typically caused by the imperfections in the system.

We suppose that the dimension of the state space is d, and that p takes on the values 0, 1, ..., d - 1. The output port for each mode is represented by a different ket  $|k\rangle$ , where k = 0, 1, ..., d - 1. Initially, all modes are present in the same input port  $|k_{in}\rangle$ , and the state vector is denoted by  $|p\rangle \otimes |k_{in}\rangle$ . To sort different modes according to their radial indices, we ensure that their output ports depend only on

their radial indices. This operation can be expressed as  $|p\rangle \otimes |k_{in}\rangle \mapsto |p\rangle \otimes |k = p\rangle$ . The successive application of a discrete Fourier transform (*F*-gate), a  $Z_d$ -gate, and a  $F^{\dagger}$ -gate can realize this transformation. The explicit transformation that each gate provides is given below:

$$\hat{F}\Big[|p\rangle \otimes |k\rangle\Big] = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp\left(\frac{i2\pi mk}{d}\right) |p\rangle \otimes |m\rangle$$

$$\hat{Z}_{d}^{j}\Big[|p\rangle \otimes |k\rangle\Big] = \exp\left(\frac{i2\pi pj}{d}\right) |p\rangle \otimes |k\rangle.$$
(2.14)

where  $\hat{F}$  and  $\hat{Z}_d$  indicates the *F*- and  $Z_d$ -gate respectively, and *j* is the order of the corresponding  $Z_d$ -gate. The  $F^{\dagger}$ -gate is the inverse *F*-gate. A  $Z_d$ -gate of order *j* is equivalent to *j* subsequent applications of the  $Z_d^1$ -gate [105].

In the first part of our implementation, we realize a binary version of our proposed radial sorter. By setting d = 2 in Eq. (2.14), the setup reduces to an interferometer with a FRFT in one of the arms. To have more control over the phase we also include a tunable phase shifter in the other arm. The  $Z_d$ -gate unit introduces a fractional Gouy phase to each of the input modes and causes distinct input modes to interfere constructively at different output ports. Thus photons of different radial indices leave the interferometer at different output ports and the sorting transformation is achieved. We note that [109] have previously demonstrated a conceptually similar design for an OAM mode sorter.

The constant phase shifter used in our experiment is a combination of waveplates, which can provide a continuously adjustable control on the phase of the travelling beam. In our setup the SLM only modulates the horizontally polarized light, thus the prepared state is horizontally polarized automatically, and the polarization state vector can be expressed as  $\mathbf{P} = [1 \ 0]^T$ , where the superscript Tmeans the transpose of a vector. The geometrical phase shifter is composed of two quarter-wave plates (QWPs) and a half-wave plate (HWP). The two QWPs are 45° oriented while the HWP between them has an angle of  $\theta$ . The Jones matrix of the geometrical phase shifter is calculated to be

$$\mathbf{M} = \frac{\sqrt{2}}{2} \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \cdot \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \cdot \frac{\sqrt{2}}{2} \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-i2\theta} & 0 \\ 0 & e^{i2\theta} \end{bmatrix}$$
(2.15)

Going through the waveplates will lead the state vector to be  $\mathbf{P}' = \mathbf{M} \cdot [1 \ 0]^T = -e^{-i2\theta}[1 \ 0]^T$ . Hence, by adjusting the angle of the HWP, we can induce an arbitrary phase shift, which is equivalent to using a piezoelectric actuator. We emphasize that our proposed sorting method is intrinsically polarization-independent, and the PBS and HWP can be replaced by the non-polarizing beam splitter (NPBS) while the geometrical phase shifter can be replaced by a piezoelectric actuator for a polarization-independent application.

The two-dimensional F-gate can be in principle realized by a NPBS. Hence, the two-dimensional sorter which consists of two quantum F-gates and a quantum Z-gate has a structure similar to the Mach-Zehnder interferometer. However, the NPBS should possess a 50/50 splitting ratio, which is not always true for the



Figure 2.3: (a) Schematics of the experimental setup. The radial mode is generated by a computergenerated hologram (CGH) on the spatial light modulator (SLM). The quarter-wave plate (QWP), half-wave plate (HWP) and QWP combination works as a geometrical phase shifter. (b, c) The measured intensity profile of the generated p = 2 and the p = 3 modes.

commercially available broadband NPBS. Here we replace the first NPBS by a HWP and a PBS. Remember that the incident beam is horizontally polarized due to the polarization sensitivity of SLM, so we can rotate the polarization angle to control the splitting ratio. After a HWP which can rotate the polarization by  $45^{\circ}$ , the two beams split by the first PBS are horizontally and vertically polarized, respectively. The second PBS will recombine the two beams after the geometrical phase shifter. However, they cannot interfere due to their orthogonal polarizations. Then we can use a HWP to rotate two beams by  $45^{\circ}$  and then split the beam by another PBS, which forms the second *F*-gate. Now the beam in each output port will have the same polarization and can interfere effectively. By aligning the rotation angle of the HWP, we can reach an arbitrary splitting ratio and can also help compensate for the possible unequal loss in two paths.

In the next step, we increase the dimensionality of the system by cascading two successive binary sorters of the type shown in Fig. 2.1(b). This configuration allows us to sort up to three radial modes. A schematic representation of our setup is depicted in Fig. 2.3. A 633 nm He-Ne laser is coupled to a single-mode fiber (SMF). The light emerging from the fiber is then collimated to illuminate an SLM. A binary computer generated hologram is imprinted onto the SLM to generate the desired field in the first diffraction order [24]. In the first stage, we use a lens with a focal length of 30 cm and with a propagation distance of z = 8.79 cm to realize a FRFT of the order  $a = \pi/4$  for a beam waist radius of  $w_0 = 207 \mu$ m. The second stage of the sorter uses two lenses with the same configuration to provide a FRFT with twice as much phase shift. We note that the interferometer shown in the schematic is imbalanced because of the need to introduce the FRFT lenses in one arm. We have taken care to keep the path imbalance much shorter than the coherence length of our laser source and the Rayleigh ranges of our modes. If one wants to sort more radial modes, more binary sorters need to be cascaded, and the order of FRFT in each stage needs to be adjusted, which is conceptually similar to the scheme shown in Fig. 4 in [109].

#### 2.3 **Results and discussions**

In order to characterize the proposed scheme, we first generate radial modes and detect the output of our setup using charge coupled devices (CCD). The images from the three CCD cameras at the three output ports of the setup are shown in Fig. 2.4(a,b). In Fig. 2.4(a) even-order modes (p = 0, 2) leave one of the output ports of the first binary sorter to CCD1 and the odd-order modes leave the

other output port. The odd-order modes are then fed into the second stage, and are routed towards CCD2, and CCD3. By changing the phase in the first stage one can send odd-order modes to CCD1 and send the even-order modes to the second stage to be sorted to CCD2, and CCD3. The cascaded binary sorters allow for sorting of up to three separate modes. As an additional test of the validity of our scheme we produce linear superpositions of three radial modes and feed it into the first stage. We then register the image of the three output ports on the CCDs simultaneously. It is clear from Fig. 2.4(c) that although all the input photons share a superposition of three radial indices, the output photons are sorted according to their radial indices. We note that to sort different sets of modes one has to choose appropriate phase differences for the two binary sorters. The value of the induced phases are different for two different sets of modes, and can be calculated using the formula for the fractional Gouy phase in Eq. (2.5). Indeed, a priori knowledge about the input state is necessary for an appropriate sorting. For any finite-dimensional sorter the input state should be restricted to a specific range.

We have quantified the crosstalk of our setup by measuring the conditional probability matrix. Each element of this matrix is defined as the probability of detecting a photon at a given mode conditioned on the radial index of the input. This quantity is equal to the power in a specific port divided by the total output power. The resulting matrix is plotted in Fig. 2.5(a). To use a single figure of merit we use the total crosstalk, which is sum of the power in the wrong ports divided by the total output power. For our specific implementation the total crosstalk is



Figure 2.4: Output port image for inputs in the form of individual LG modes and their superposition states. The position of each CCD is shown in Fig. 2.3. (a) The path lengths in the first stage are adjusted so the even-order modes are sent to CCD1 whereas the odd-order modes are sent to the second stage where they are further sorted so that p = 1 (p = 3) is directed to CCD2 (CCD3). (b) The phase shifter in the first stage is readjusted to send odd-order modes to CCD1 and the even-order modes to the second stage. (c) The images on CCDs when a superposition state is sent to the sorter. p = 0, 1, 2 means that a superposition state composed of p = 0, p = 1 and p = 2 mode is generated and injected. All images in the same line are captured simultaneously.

measured to be 15%. In addition we wish to emphasize that this crosstalk is not intrinsic to the protocol. We believe that using high-quality anti-reflection coated optics, active stabilization, and more careful alignment can mitigate crosstalk significantly and bring the sorter to its theoretical limit of 100% efficiency and no crosstalk.

As mentioned above, our scheme can also be used for sorting of photons according to their OAM number. To demonstrate this capability we use the first stage of our setup to implement a binary sorting of  $LG_{10}$  and  $LG_{12}$ . The images of the output ports are plotted in Fig. 2.5(b), and confirm that photons of different OAMs leave the interferometer at separate ports. We underscore the fact that here we have separated two OAM modes of the same radial order whose OAM values



Figure 2.5: (a) Experimentally measured probability of detection as a function of input and output mode indices. (b) The measured output intensity profile for an input prepared as a superposition of  $p = 1, \ell = 0$  and  $p = 1, \ell = 2$  modes.

are different by  $\Delta \ell = 2$ . The spacing by two units results from the fact that the phase shift from the FRFT is  $\Delta \phi = (2p + |\ell|)a$ . The extra factor 2 for p index implies that the  $\ell$  spacing has to be twice larger. Of course, by selecting the appropriate order of the FRFT, our device can sort the LG beams with  $\Delta \ell = 1$  as well.

Compared to the multi-channel interferometer proposed in [101], our cascading scheme is advantageous in terms of flexibility, complexity and practicality [1]. In this section, we compare the cascading interferometer presented in the manuscript and the multi-channel interferometer proposed in [101] in terms of complexity, flexibility and practicality. An advantage of the cascaded scheme is that it has a simpler structure for high dimensional system. In the multi-channel interferometer shown in the Fig. 2.1, there are two quantum F-gates which can be implemented by the 50/50 beam splitters. For a  $2^n$ -dimensional sorter, the first quantum F-gate is effectively a 1-to- $2^n$  splitter and it requires  $2^n - 1$  beam splitters. The second quantum F-gate, however, needs  $(n+2) \cdot 2^{n-1} - 1$  beam splitters and the calculation is as follows. Assume a d-dimensional F-gate requires  $N(F_d)$  beam splitters. Then we will have the following recursive relation [103]

$$N(F_{2d}) = 2 \cdot N(F_d) + d \tag{2.16}$$

With the knowledge that  $N(F_2) = 1$ , we can arrive at the following equation  $N(F_d) = (d/2) \cdot \log_2 d$ . Setting  $d = 2^n$ , we will have  $N(F_{d=2^n}) = n \cdot 2^{n-1}$ . So the total number of beam splitters is  $n \cdot 2^{n-1} + 2^n - 1 = (n+2) \cdot 2^{n-1} - 1$  for the multi-channel interferometer. As for the cascading interferometer presented in our work, we will need  $2^n - 1$  Mach-Zenhder interferometer (MZI) to sort  $2^n$  modes, and each MZI requires two beam splitters. So the total number is  $2 \cdot (2^n - 1) = 2^{n+1} - 2$ . It can be readily verified that  $2^{n+1} - 2 < (n+2) \cdot (n+2)$  $2^{n-1}-1$  for  $n \ge 2$ . In addition, a multi-channel interferometer requires that all paths remain in-phase simultaneously. This can cause a serious challenge for experimental realization as any phase mismatch will compromise the functionality of all output channels. In contrast, if one of the sorter paths in the cascading scheme is not in-phase, the system continues to properly operate for the channels that do not utilize the mismatched path. Furthermore, the conventional quantum F-gate is usually composed of 50/50 beam splitters and phase shifters [103], and thus it always has a dimensionality of  $d = 2^n$ , where n is an integer. Thus it would be difficult to realize a sorter with a dimensionality that is not a power of 2. Taking the first quantum *F*-gate as an example, one would need to use 33/67 beam splitters to build a d = 3 system, and 20/80 beam splitters for d = 5 system. Considering this, the cascaded approach is simpler to realize due to the commercial availability of 50/50 beam splitters.

We note that our design can also be employed for sorting the HG modes. Coherent detection of LG and HG modes has been recently identified as an optimal means of localizing closely spaced incoherent sources [98, 110–112]. It is thus reasonable to expect that an efficient sorting mechanism can have further implications for microscopy, given the significance of super-resolution in that field. In addition, a similar approach can be applied to sorting the family of Bessel-vortex beams. Due to the non-diffracting property of these modes, free-space propagation can serve as the Z-gate and there is no need for realization of the FRFT module. Hence, a simplified version of our experiment with the the FRFT components removed would be able to sort Bessel beams with different longitudinal wavevectors.

In summary we have demonstrated a general framework for efficient measurement (i.e., sorting) of the radial index of LG modes. Our protocol includes two essential elements: the discrete Fourier transform (*F*-gate) and the  $Z_d$ -gate. While discrete Fourier transform can be realized using beam splitters and wave plates, we have employed the fractional Gouy phase to realize the  $Z_d$ -gate efficiently. As a demonstration we have implemented a binary (d = 2) version of our protocol and have cascaded two binary sorters to sort three different LG modes according to their radial indices. We believe that implementation of our protocol can facilitate fundamental studies of the spatial modes of light as well as a variety of prevalent applications of such states in quantum communications, imaging, and quantum metrology [113].

### **Chapter 3**

# **Realization of a Hermite-Gauss mode sorter**

The transverse modes of electromagnetic waves have long been used in fundamental studies of beam propagation [9]. Spatial mode decomposition of optical fields can facilitate the understanding and analyzing of optical beam in free space [114, 115], graded-index MMFs [66, 116], and waveguides [117]. Beyond their advantage for theoretical work, the transverse degrees of freedom of photons are recognized as information resources for both classical and quantum information technologies because of the unbounded Hilbert space spanned by these spatial mode basis sets [22, 118]. The HG modes are the propagation-invariant modes in parabolic-index MMFs [66, 119] and closely resemble the communication modes of square apertures for free-space propagation [120], which suggests the potential of HG modes in optical communications. In addition, very recently it has been shown that the detection of HG modes can beat "Rayleigh's curse" and realize super-resolution imaging [98]. It is proposed that spatial mode decomposition in the HG basis can reach the Cramér-Rao bound for resolving two closely located point sources, while the classical Fisher information of traditional imaging in the position basis inherently drops to zero. There have been various experimental investigations that propose alternative detection strategies to achieve a nonzero Fisher information for a small separation [110–112]. However, because of the lack of an efficient HG mode sorter, the Cramér-Rao bound remains to date inaccessible.

Despite the usefulness of HG modes in various areas, practical applications have been so far impeded by the difficulty in efficient detection of these modes. In contrast, the sorting of the LG modes, the rotationally-symmetric counterparts of HG modes, have been experimentally demonstrated [43, 109, 121]. The azimuthal structure of LG modes is found to be directly related to the OAM of photons [13], and it has been shown that a Cartesian to log-polar transformation can enable an OAM mode sorter to identify a large number of OAM modes [43, 91, 121]. Nonetheless, this method is not directly applicable to the HG modes. Several previous investigations on the sorting of HG modes are based on cascaded Mach-Zenhder interferometers [122, 123], and the practicality of such approaches greatly limits the number of HG modes that can be sorted. The multi-plane light conversion method has also been proposed to sort the spatial modes [124]. While this method is able to separate many modes simultaneously, the crosstalk between neighboring modes is relatively high, and a large number of phase planes are usually needed to reduce the crosstalk. Therefore it remains highly desirable to build a robust sorter that can efficiently separate many HG modes with low crosstalk.

#### 3.1 Astigmatic mode conversion

Here we describe a method for efficiently sorting HG modes. This method entails converting a HG mode to a unique LG mode and then sorting the LG mode with known methods. This sorter operates by sequentially applying a fractional Fourier transform (FRFT) module [1], an astigmatic mode converter [104, 125], and an OAM mode sorter. Our scheme takes advantage of a useful relation between HG modes and LG modes: the conversion between these two families of modes can be realized by an astigmatic mode converter [104]. This mode converter can be implemented by two cylindrical lenses, and it has been shown that such a converter can transform  $HG_{mn}$  to  $LG_{p\ell}$  conditioned on  $p = \min(m, n)$  and  $\ell = m - n$ , where m and n are the mode indices of a HG mode along x and y directions, and p and  $\ell$  are the radial and azimuthal indices of a LG mode. Therefore, when we cascade an astigmatic mode converter and an LG mode sorter, we can realize a HG mode sorter. In Fig. 3.1(a) the conversion relationship between HG modes and LG modes is visualized, and the experimental results of two HG-LG mode conversions are shown in Fig. 3.1(b). The HG modes in different columns in Fig. 3.1(a) can be converted to LG modes of different OAM indices, and therefore can be resolved by an OAM sorter. For example, we note that  $HG_{m0}$  (HG<sub>0n</sub>) can be transformed to  $LG_{0m}$  ( $LG_{0,-n}$ ), and therefore one can readily utilize an OAM mode sorter to unambiguously sort HG modes with one zero index. This result leads to significant consequences, such as for super-resolution imaging in microscopy, as an efficient sorting of HG<sub>00</sub>, HG<sub>01</sub>, and HG<sub>10</sub> modes can resolve



Figure 3.1: (a) Conversion relation of HG modes and LG modes. Each dot represents a  $HG_{mn}$  mode and the number under the dot denotes the mode indices mn. The HG modes in the same column are converted to LG modes of the same OAM index  $\ell$ , and the modes in the same row correspond to the LG modes of the same radial index p. (b) Experimental results of mode conversion from  $HG_{20}$  to  $LG_{02}$  and from  $HG_{42}$  to  $LG_{22}$ .

sub-diffraction objects and attain the Cramér-Rao bound [98].

In order to determine both mode indices m and n simultaneously, however, the knowledge of  $\ell = m - n$  is not sufficient. In other words, another index  $p = \min(m, n)$  needs to be identified to completely characterize the LG mode converted from a HG mode. Here we resort to the FRFT as a secondary sorting mechanism, which has been recently used to sort the radial index of LG modes [1, 3]. Since the HG modes are the eigenmodes of FRFT, they keep invariant under such a transform except for a mode-dependent Gouy phase. This transformation can be written as [105]

$$\mathcal{F}^{a}[\mathrm{HG}_{mn}(x_{0}, y_{0})] = \exp[-i(m+n)a] \cdot \mathrm{HG}_{mn}(x, y), \tag{3.1}$$

where  $\mathcal{F}^a$  denotes an FRFT of order *a*, and  $HG_{mn}(x, y)$  is the transverse field distribution at the beam waist plane. One can notice that the fractional Gouy phase  $\exp[-i(m+n)a]$  contains information regarding the mode order m + n, and it has been demonstrated that an interferometer can be built to sort beams of different mode order m + n to different output ports [122, 123]. In our experiment, we implement a common-path interferometer by using polarization-dependent SLMs to realize an inherently stable sorter [2, 3]. The horizontal and vertical polarizations are employed as two arms of a Mach-Zenhder interferometer, and two polarization-sensitive SLMs are used to perform FRFT of different orders to the two polarizations respectively. If the FRFT order difference is  $\Delta a$ , then a modedependent phase of  $\Delta \phi = -(m+n) \cdot \Delta a$  is introduced between the two polarizations at the output of the FRFT module. By measuring the polarization state of the output photons, one can determine the value of m + n. Due to the bounded twodimensional Hilbert space of polarization, our FRFT module and the following OAM mode sorter can only resolve the HG modes located at the top two rows in Fig. 3.2(a). However, the sorting capability can be readily extended by cascading more FRFT modules as will be discussed later.

#### **3.2** Experimental realization

The experimental schematic of our HG mode sorter is shown in Fig. 3.2 . A 633 nm He-Ne laser is spatially filtered by a SMF and then collimated to illuminate SLM 1. A computer-generated hologram is imprinted on SLM 1 to generate HG modes in the first diffraction order [34]. A polarizer sets the light to be diagonally polarized. The beam waist radius of HG modes is 462.3 µm. A quadratic



Figure 3.2: Schematic of the HG mode sorter. A binary grating on SLM1 generates HG modes in the first diffraction order that are then analyzed by the rest of the setup. The FRFT module consists of two SLMs and two lenses. Two cylindrical lenses (CLs) form the  $\pi/2$  mode converter. The OAM sorter consists of an unwrapper and a phase corrector to perform Cartesian to log-polar transformation.

phase equivalent to that of a 0.62 m lens is imprinted on SLM 2 and SLM 3. Each SLM is attached with a lens of focal length 1.5 m, and in the experiment we realize this by relaying each lens to the corresponding SLM via a 4-*f* system respectively. Both SLMs are only effective to horizontal polarization and do not modulate the vertical polarization. The free-space propagation distance between SLM 2 and SLM 3 is chosen to be 2z, where z = 0.44 m. It can be verified that SLM 2, SLM 3, and two spherical lenses implements an FRFT of  $a = \pi/2$  to vertical polarization [3]. For horizontal polarization, these elements act as a 4-*f* system of  $a = \pi$ . According to Eq. (2.5), the diagonally polarized input HG modes become diagonally (anti-diagonally) polarized at the output when the value (m + n)/2 is

even (odd). After this module, two relay lenses resize the beam waist radius to match an astigmatic mode converter constituted by a pair of 45-degree oriented cylindrical lenses. The focal length of each cylindrical lens  $f_{CL}$ , the beam waist radius  $w_0$ , and the separation between two cylindrical lenses  $s_{CL}$  are related by the following equations [104]:

$$w_0 = \sqrt{\frac{(1+1/\sqrt{2})\lambda f_{CL}}{\pi}}, \qquad s_{CL} = \sqrt{2}f_{CL},$$
 (3.2)

where  $\lambda = 633$  nm is the laser wavelength. In our experiment we use  $f_{CL} = 10$  cm and  $s_{CL} = 14.1$  cm, and the consequent beam waist radius is  $w_0 = 185.5$  µm. The transformed HG modes, which have become LG modes at this point, are sent to a polarization-independent OAM mode sorter [121]. A HWP and a PBS directs photons to different output ports according to their orthogonal polarization assigned by the FRFT module.

#### **3.3 Results and discussions**

The experimental results of our HG mode sorter are shown in Fig. 3.3. In Fig. 3.3(ad) we present the images on two cameras when HG modes are injected into the sorter individually. In Fig. 3.3(e) we combine the sorting result of  $HG_{0n}$  with  $n = 0, 2, \dots, 8$  as well as  $HG_{m0}$  with  $m = 0, 2, \dots, 8$ . The lowest index of these modes is 0, which is represented by the first row in Fig. 3.1(a). We choose the index spacing to be 2 to reduce the overlap between neighboring modes for



Figure 3.3: Experimental results of the HG mode sorter. (a)-(d) Intensity profile on two cameras when individual HG modes are injected. The mode indices mn are labelled beside each sorted mode. (e) Combined intensity profile on two cameras when incident modes include HG<sub>0n</sub> of  $n = 0, 2, \dots, 8$  and HG<sub>m0</sub> of  $m = 0, 2, \dots, 8$ . (f) Combined intensity profile at two output ports when incident modes include HG<sub>1n</sub> of n = 1, 3, 5, 7 and HG<sub>m1</sub> of m = 1, 3, 5, 7.

the purpose of better visualization. We emphasize that there is no fundamental restriction on the index spacing provided that the OAM mode sorter can have a sufficient mode resolution [43]. One can notice that the HG modes for which (m+n)/2 is even (odd) are routed to camera 1 (camera 2) as a result of the FRFT module. As mentioned earlier, one FRFT module and one OAM mode sorter can resolve the modes in the top two rows in Fig. 3.1(a), and the experimental evidence is presented in Fig. 3.3(f). We generate HG<sub>1n</sub> with n = 1, 3, 5, 7 and HG<sub>m1</sub>

with m = 1, 3, 5, 7, and combine the corresponding images from two cameras. The lowest index of these modes is 1, which are represented by the second row in Fig. 3.1(a). Notably, one can see that these two sets of modes occupy different positions on cameras and in principle can be fully identified by a high-resolution OAM sorter [43]. However, we also note that our setup cannot separate HG modes whose mode order m + n is odd. For example, one can verify that HG<sub>10</sub> and HG<sub>21</sub> cannot be separated by either FRFT module or the subsequent OAM mode sorter. This problem can be addressed by cascading a Dove prism and a Sagnac interferometer to the FRFT module [126]. Here we provide a conceptual design and demonstrate how to extend the dimension of a HG mode sorter. As shown in Fig. 3.4(a), the input linearly polarized  $HG_{mn}$  is converted to  $LG_{p\ell}$  and sent to the subsequent LG mode sorter. Compared to Fig. 3.2, we place the astigmatic mode converter before the FRFT module due to the consideration of simplicity. The input  $HG_{mn}$  mode and the converted  $LG_{p\ell}$  mode is related by  $m + n = 2p + |\ell|$ , and the phase induced by FRFT is a function of m + n for HG modes and  $2p + |\ell|$ for LG modes. Therefore swapping the mode converter and the FRFT module does not influence the sorting mechanism. Each FRFT module is followed by a Dove prism inside a Sagnac interferometer. The FRFT module induces a phase difference  $\Delta \psi_1 = -(m+n) \cdot \Delta a$  to horizontal and vertical polarizations [1], and the Dove prism can rotate horizontally and vertically polarized LG modes by an angle of  $2\beta$  and  $-2\beta$  respectively, where  $\beta$  is the orientation angle of the Dove prism. This polarization-dependent rotation leads to a phase difference of  $\Delta \psi_2 = -4\beta(m-n)$  between horizontal and vertical polarizations [109], where

 $m-n=\ell$  represents the OAM index of the converted LG modes. Therefore the total phase difference is  $\Delta \psi = -[(\Delta a + 4\beta)m + (\Delta a - 4\beta)n]$  and one can realize an unambiguous sorting with appropriately selected  $\Delta a$  and  $\beta$ . These parameters used in each stage are provided in Fig. 3.4(a). It can be verified that all HG modes will now be guided towards a unique, mode-dependent output port [126], and the output port of each mode can be predicted by Eq. (2.5). We numerically simulate the output image of this extended sorter, and the combined simulation result for 20 HG modes is presented in Fig. 3.4(b). These 20 HG modes are the modes listed in Fig. 3.1(a) with a mode index spacing of 1. In the simulation we use a beam-copying grating in an OAM sorter, which can create 7 copies of a beam to improve the mode resolution [43]. It can be noticed that these 20 HG modes can be well separated to distinct positions unambiguously with negligible crosstalk. As demonstrated in [43], with a beam-copying grating the separation efficiency can be improved to theoretically 97%, and experimentally achieved efficiency can be larger than 92%. We also emphasize that by removing the astigmatic LG-HG mode converter, this sorter becomes a LG mode sorter and can unambiguously separate LG modes of  $p \in \{0, 1, 2, 3\}$  and arbitrary  $\ell$ .

In our experiment the loss mainly comes from the SLMs due to the limited light utilization efficiency, which might impede the scaling shown in Fig. 3.4. However, we note that the SLMs can be readily replaced by other low-loss devices, such as the commercially available polarization directed flat lenses [126]. In addition, since all devices employed in our scheme are essentially phase-only elements, the loss can in principle be reduced to zero if appropriate anti-reflection



Figure 3.4: (a) Conceptual schematic of HG mode sorter of an extended dimension. A HG<sub>mn</sub> is first transformed to  $LG_{p\ell}$ , and then FRFT modules cascaded by a Dove prism inside a Sagnac interferometer are applied to sort the LG mode. Relay lenses to prevent spatial modes from diffraction are omitted for simplicity. (b) Simulated sorting result for entire 20 HG modes which are listed in Fig. 3.1(a). Each mode is normalized to have the same maximum intensity. It can be noticed that each mode is mapped to distinct position on cameras unambiguously. The mode index mn is labeled around the corresponding sorted mode.

coating is applied on all elements. However, given that the polarization has been used in our scheme to realize a robust FRFT module, our sorter cannot work directly for an arbitrary polarization. This limitation can be lifted e.g., by using a polarization-independent FRFT module [1] or by inserting a PBS to separate polarizations before the sorter. Moreover, the sorting scheme presented here can in principle be used to HG modes of different wavelengths. As can be seen in Eq. (3.2), the parameters  $f_{CL}$  and  $s_{CL}$  remains the same as long as  $w_0^2/\lambda$  keeps constant, which is also true for the parameters of the FRFT module [see Eq. (2.13)]. We also note that the material dispersion is the only factor that limits the spectral bandwidth of OAM mode sorter [121]. Therefore, our sorter in principle can have a relatively broad bandwidth provided that  $w_0^2/\lambda$  is constant and material dispersion is small.

In this work, we use a spatially coherent laser as the light source to test the performance of our HG mode sorter. It should be noted that a mode sorter can also be used to measure the HG mode spectrum of spatially incoherent beams. It has been shown that a HG mode sorter can be used to efficiently resolve the separation between two incoherent point sources [98]. However, the coherences (i.e., off-diagonal elements in a density matrix) between different modes cannot be directly measured by a HG mode sorter. Quantum state tomography is the conventional method used to characterize density matrices for general quantum states. However, the data acquisition time generally scales linearly with the dimension of the Hilbert space, hindering the possibility of dynamic monitoring of a high-dimensional quantum system. Recently, a direct tomography protocol has been demonstrated to measure density matrices in the spatial domain [127], which has been applied to characterize the density matrix of an incoherent mixture of HG modes.

In conclusion, we have proposed and experimentally realized a scalable scheme that can efficiently sort a large number of HG modes. Our scheme is based on an astigmatic mode converter to transform HG modes to LG modes and takes advantage of a LG mode sorter to realize a mode sorter for a large number of HG modes. Further increasing the dimension is straightforward and a conceptual schematic has been presented and numerically simulated. Taking into account the broad use of HG modes, we expect that our demonstration can facilitate fundamental studies of beam analysis in free space and graded-index multi-mode fiber, and can enhance a variety of applications such as QKD, super-resolution imaging, and classical communications.

#### **Chapter 4**

# Quantum key distribution with high-dimensional spatial mode encoding

Within the past few decades the OAM modes have received extensive attention [37, 64, 83, 128, 129] and are widely applied in various information technologies, including quantum teleportation [130], optical communications [22], and QKD [19, 20, 131]. Compared to the intrinsically bounded polarization state space, the OAM modes offer an infinite-dimensional Hilbert space for information encoding and therefore can be used to increase the transmission rate of a communication link. However, the finite size of apertures in a realistic system usually constrains the dimension of OAM state space that can be accessed. On the other hand, the OAM modes only account for azimuthal variations in the transverse plane, and it has been shown that these modes cannot reach the capacity limit of a communications link without including the radial degree of freedom [94]. It is thus highly desirable to multiplex the remaining radial degree of freedom to increase the transmission rate for both classical and quantum communications.

A complete and orthonormal basis that incorporates both radial and azimuthal variations can be constituted by LG modes. The LG modes are characterized by a radial quantum number p and an OAM quantum number  $\ell$ , and arbitrary paraxial field can be described by these modes [13]. Notably, LG modes have been shown to be good approximations to the eigen propagation modes of circular apertures [120], and their small divergence angle and intrinsic rotational symmetry make these modes preferable in free-space optical communications. Moreover, it is apparent that the transmission rate of a communications system can be further increased by using both azimuthal and radial degrees of freedom, and it has been shown that the radial index p can potentially mitigate the power loss when the receiver has a limited aperture size [132]. In addition, different from the vortex phase structure related to the OAM index, the radial index corresponds to a radial, amplitude-only distribution. While the phase structure can be strongly distorted by atmospheric turbulence [133], the intensity pattern of the transmitted beam can remain recognizable for a 1.6-km free-space link [128], and an intensity pattern recognition accuracy can be higher than 98% for a 3-km link [52] and 80% for a 143-km link [134], which suggests that the amplitude structure associated with a nonzero radial index may be helpful to turbulence mitigation [118]. Recent advances have shown how to efficiently measure and use p and  $\ell$  in optical communications [86, 91, 126, 135], but the mode sorting in mutually unbiased basis of LG modes required by a QKD protocol has not previously been reported. While a deterministic detection scheme for mutually unbiased bases of polarization and OAM degrees of freedom has been developed [136], the same strategy cannot be

directly applied to spatial modes due to the lack of efficient manipulation devices such as HWPs for spatial modes. Besides the usefulness in QKD, the capability of sorting superposition modes can be helpful to other quantum applications such as certifying high-dimensional entanglement [137] as well as super-resolution imaging [138]. In the following we present how to build a superposition mode sorter and provide the experimental demonstration of a QKD protocol employing all three transverse degrees of freedom.

#### 4.1 Construction of mutually unbiased bases

To develop a BB84 protocol one needs at least two sets of bases that are mutually unbiased: Every element in each basis is a uniform superposition of elements in another basis, which guarantees that measurement in the wrong basis reveals no information of the measured state. Here we first examine each transverse degree of freedom individually and then demonstrate how to construct two mutually unbiased bases for QKD. The first degree of freedom employed in QKD protocol is the polarization of photons [139], which provides a two-dimensional state space spanned by horizontal  $|H\rangle$  and vertical  $|V\rangle$  polarizations. While polarization is the most widely used degree of freedom in QKD, the azimuthal structures of light have recently seen increasing usage due to their direct relation with the OAM of photons [19]. Each photon can carry an OAM of  $l\hbar$  [13] and the corresponding OAM state can be written as  $|l\rangle_{\ell}$ , where l is an arbitrary integer and the subscript  $\ell$  denotes an OAM state. In this work we restrict ourselves to an OAM subspace



Figure 4.1: Intensity profiles of elements in the main and conjugate bases. The normalized intensity distribution of each spatial mode is presented. The upper right image is the corresponding experimental (Exp.) record of the simulated (Sim.) image for comparison. Each mode can have the two different polarizations as shown at the upper left corner. Scale bar, 1 mm.

that is spanned by  $|\pm 2\rangle_{\ell}$ . Finally we use the radial quantum number p of LG functions as another independent resource for information encoding [95, 140]. Again we restrict ourselves to a subspace that is spanned by the two lowest radial indices p = 0, 1. The direct product of these three subspaces constitutes an 8-dimensional Hilbert space that is used as our first basis for QKD. The elements in this basis are  $\{|H, 0_p, -2_{\ell}\rangle, |H, 0_p, 2_{\ell}\rangle, \cdots, |V, 1_p, 2_{\ell}\rangle\}$ . We refer to this basis as the *main* basis. The above-specified choices of bases for azimuthal and radial degrees of freedom allows us to employ recent advances in sorting LG modes according to both OAM and radial indices [1, 43, 91, 141].

To implement secure QKD we need an additional capability of measuring pho-

tons in at least one conjugate basis. The conjugate basis is mutually unbiased with respect to the main basis, and can be constructed through a direct product of respective complementary bases of polarization, azimuthal, and radial subspaces [142]. The complementary basis of polarization subspace can be built as

$$|D\rangle = \frac{|H\rangle + |V\rangle}{\sqrt{2}}, \qquad |A\rangle = \frac{|H\rangle - |V\rangle}{\sqrt{2}}, \qquad (4.1)$$

where  $|D\rangle$  and  $|A\rangle$  denote the diagonal and anti-diagonal polarizations respectively. We use a similar choice for OAM states and define

$$|\ell_D\rangle = \frac{|-2\rangle_\ell + |2\rangle_\ell}{\sqrt{2}}, \qquad |\ell_A\rangle = \frac{|-2\rangle_\ell - |2\rangle_\ell}{\sqrt{2}}, \qquad (4.2)$$

The complementary basis for radial modes is taken to be

$$|p_L\rangle = \frac{|0\rangle_p + i|1\rangle_p}{\sqrt{2}}, \qquad |p_R\rangle = \frac{|0\rangle_p - i|1\rangle_p}{\sqrt{2}}, \qquad (4.3)$$

where the subscript L and R follow the notation of left- and right-handed circular polarization. We choose this definition because such states are easier to generate experimentally, but we stress that this will not make any fundamental change to the QKD protocol. With all these definitions, the elements in the complementary basis, referred to as *conjugate* basis, are  $\{|D, p_L, \ell_D\rangle, |D, p_L, \ell_A\rangle, \dots, |A, p_R, \ell_A\rangle\}$ . Spatial intensity profiles of all these modes are given in Fig. 4.1.

Having established the two mutually unbiased bases, we need to perform coherent detection in each of these bases. For the main basis, devising a coherent detection strategy is straightforward. A sequence of a polarizing beamsplitter (PBS), a radial mode sorter [1], and an OAM sorter [91] can losslessly project input photons onto elements in the main basis. However, each state in the conjugate basis is a superposition of different LG modes and cannot be efficiently measured by using radial and OAM sorter only. To address this problem, we develop a generic, scalable scheme for sorting such superposition states and experimentally realize it as a part of the QKD protocol. For simplicity we only focus on the radial degree of freedom thereafter, and the same scheme can be directly applied to other degrees of freedom.

A conceptual schematic for a radial superposition mode sorter is shown in Fig. 4.2. For a *d*-dimensional Hilbert space spanned by radial modes  $|m\rangle_p$ , where  $m \in \{0, 1, \dots, d-1\}$ , the corresponding complementary basis can be defined as

$$\left|\bar{n}\right\rangle_{p} = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \exp\left(\frac{-i2\pi m\bar{n}}{d}\right) \left|m\right\rangle_{p}, \qquad (4.4)$$

where  $\bar{n} \in \{0, 1, \dots, d-1\}$  is the superposition mode index, and a bar above the number indicates that this is an element in the complementary basis. The port of each state is labelled by another distinct ket  $|k\rangle$ , where  $k \in \{0, 1, \dots, d-1\}$ . Initially, all superposition modes are located at the same port, which can be expressed as  $|\bar{n}\rangle_p |0\rangle$ . We first apply a radial mode sorter to direct different radial mode components towards distinct, non-overlapping ports [1], which can



Figure 4.2: Schematic of the generic radial superposition mode sorter. (a) Conceptual schematic of a *d*-dimensional radial superposition mode sorter. (b) Two-dimensional (d = 2) realization to sort  $|p_L\rangle$  and  $|p_R\rangle$ . The unitary transformation  $\hat{U}_{1\to0}$  is realized by two SLMs. The modes  $|p_L\rangle$  and  $|p_R\rangle$  are directed to different output ports as indicated at the last beamsplitter.

be expressed as

$$\left|\bar{n}\right\rangle_{p}\left|0\right\rangle = \frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\exp\left(\frac{-i2\pi m\bar{n}}{d}\right)\left|m\right\rangle_{p}\left|0\right\rangle \to \frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\exp\left(\frac{-i2\pi m\bar{n}}{d}\right)\left|m\right\rangle_{p}\left|m\right\rangle$$

$$(4.5)$$

Then a mode converter performs a unitary transformation  $\hat{U}_{m\to 0}$  to convert individual radial modes  $|m\rangle_p$  to the same state  $|0\rangle_p$ , which enables effective interference between these otherwise orthogonal modes and the consequent state becomes

$$\frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\exp\left(\frac{-i2\pi m\bar{n}}{d}\right)\left|m\right\rangle_{p}\left|m\right\rangle \to \frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\exp\left(\frac{-i2\pi m\bar{n}}{d}\right)\left|0\right\rangle_{p}\left|m\right\rangle.$$
(4.6)

Finally a discrete Fourier transform performed by a quantum F-gate can direct photons to different output port indexed by the superposition mode index  $\bar{n}$  as [101]

$$\hat{F}\left[\frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\exp\left(\frac{-i2\pi m\bar{n}}{d}\right)|0\rangle_{p}|m\rangle\right] = \frac{1}{\sqrt{d}}\sum_{m=0}^{d-1}\exp\left(\frac{-i2\pi m\bar{n}}{d}\right)\hat{F}\left[|0\rangle_{p}|m\rangle\right]$$
$$= \frac{1}{d}\sum_{j=0}^{d-1}\sum_{m=0}^{d-1}\exp\left(\frac{i2\pi m(j-\bar{n})}{d}\right)|0\rangle_{p}|j\rangle = \sum_{j=0}^{d-1}\delta(j-\bar{n})|0\rangle_{p}|j\rangle = |0\rangle_{p}|\bar{n}\rangle,$$
(4.7)

where the operation of quantum F-gate is defined as

$$\hat{F}\left[\left|m\right\rangle_{p}\left|k\right\rangle\right] = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\exp\left(\frac{i2\pi jk}{d}\right)\left|m\right\rangle_{p}\left|j\right\rangle.$$
(4.8)

Therefore, through the use of a mode converter, the superposition mode  $|\bar{n}\rangle_p$  can be efficiently sorted to  $\bar{n}$ -th output port with in principle unity efficiency and zero crosstalk.

A two-dimensional conceptual realization of this scheme to sort  $|p_L\rangle$  and  $|p_R\rangle$  is shown in Fig. 4.2(b) as an intuitive example, where the binary radial mode sorter follows the scheme in [1] and is realized by a Mach-Zenhder interferometer with extra lenses in one arm to perform the Fourier transform. After the radial mode sorter, the radial modes  $|0\rangle_p$  and  $|1\rangle_p$  are sorted to different paths, but the relative

phase determined by the definition of  $|p_L\rangle$  and  $|p_R\rangle$  as in Eq. (4.3) persists. The relative phase is represented by the factor  $\pm i$  in Fig. 4.2(b). The subsequent mode converter realized by SLM transforms the LG mode  $|1\rangle_p$  to  $|0\rangle_p$  [40, 143], and the phase imparted to the SLMs can be calculated by a nonlinear fitting algorithm [144]. Then a beamsplitter recombines the modes and acts as a binary quantum F-gate [101]. Before the final beamsplitter, the incident modes at the input ports can be written as  $|0\rangle_p$  and  $\pm i |0\rangle_p$  respectively, where the factor  $\pm i$  represents the relative phase between the initial  $|0\rangle_p$  and  $|1\rangle_p$  states. Due to the interference at the beamsplitter, the sign of the phase term  $\pm i$  determines the port that the photon will be directed into. Therefore, through the use of a radial mode sorter, a mode converter realized by the SLMs, and a binary quantum F-gate realized by a beamsplitter,  $|p_L\rangle$  and  $|p_R\rangle$  can be efficiently separated to different output ports. We note that this scheme should also be applicable to other spatial modes such as HG modes given the existence of a corresponding mode sorter [145] and converter [143], which can be useful to realize super-resolution imaging [138].

## 4.2 High-dimensional spatial mode encoding and detection based on a generic mode sorter

In our experiment, we use polarization-sensitive SLMs to develop a stabilized, common-path radial mode sorter and converter as shown in Figs. 4.3 and Fig. 4.4, respectively. The common-path radial mode sorter is composed of two polarization-sensitive SLMs, each with a spherical lenses attached respectively. Each spherical


Figure 4.3: Experimental realization of the common-path radial mode sorter.

lens is imaged onto the respective SLM by a 4-f system in the experiment. A quadratic phase pattern which is equivalent to a lens of focal length 0.62 m is imprinted on both SLMs. Both spherical lenses have a focal length of  $f_1 = 1.5$  m. The distance between the injected mode beam waist plane and the first SLM is z = 0.44 m, and the separation between two SLMs is 2z. The output plane of the radial mode sorter is of distance z after the last SLM. The Fourier transforming lens in radial mode converter has a focal length of 1 m. In this setup two polarizations are employed as two arms of a Mach-Zenhder interferometer, and the injected mode is 45 degree polarized. Since vertically polarized light is not affected by SLMs, one can check that two spherical lenses perform a Fourier transform to it [1]. Horizontal polarization, however, is modulated by both SLMs. Each SLM with the attached lens becomes a lens of focal length 0.44 m, exactly equal to the propagation distance z. Hence, horizontally polarized light experiences two consecutive Fourier transforms. If we treat vertically polarized beam as the ref-



Figure 4.4: Experimental realization of a radial mode converter which transforms  $|1_p, 2_\ell\rangle$  to  $|0_p, 2_\ell\rangle$ .

erence arm, then horizontally polarized beam will gain a mode-dependent Gouy phase of  $\exp(-ip\pi)$  due to its extra Fourier transform. One can check that this phase is 0 for  $|0\rangle_p$  and  $-\pi$  for  $|1\rangle_p$  [1], and thus  $|0\rangle_p$  remains 45 degree polarized, but the polarization of  $|1\rangle_p$  is rotated to 135 degrees. In the experiment we use a half-wave plate (HWP) so that  $|0\rangle_p$  is vertically polarized while  $|1\rangle_p$  becomes horizontally polarized at the output port. Compared to the sorter based on Mach-Zenhder interferometer, this common-path interferometer is robust to vibration and air turbulence. The beam waist radius of LG modes used in our experiment is  $w_0 = 462.3 \ \mu$ m, and we note that the parameters of the radial mode sorter mentioned above are specific to this beam waist radius and cannot be directly applied to LG modes of different beam size [1]. However, one can always use a 4-*f* system with appropriate lateral magnification to adjust the beam waist radius. In addition, the OAM index  $\ell$  of the incident LG mode can also affect the radial



Figure 4.5: (a) The phase profile as a function of radius r on the two SLMs in the radial mode converter. (b) The input and output spatial mode intensity profile of the radial mode converter. The upper right image is the corresponding experimental (Exp.) record of the simulated (Sim.) image for comparison.

mode sorter because the Gouy phase is also a function of  $|\ell|$ . In this experiment we use  $\ell = -2$  and  $\ell = 2$ , so the value of  $|\ell|$  is a constant and thus its effect can be ignored. To remove the  $\ell$ -dependence of radial mode sorter, one can use a Dove prism to cancel the  $\ell$ -dependent phase as demonstrated in [126].

The common-path radial mode converter is implemented by two SLMs connected by a Fourier transforming lens to realize mode conversion from  $|1_p, \pm 2_\ell\rangle$ to  $|0_p, \pm 2_\ell\rangle$ . Taking into account the fact that the converter only reshapes the radial structure of the mode, the phase written on SLMs should be a function of radius r only. So the phase on first SLM can be decomposed to the polynomials

of radius and we use nonlinear fitting algorithm [144] to improve the conversion efficiency by adjusting coefficients of these polynomials. The second SLM is used to cancel the residue phase of the converted mode because  $|0_p,\pm 2_\ell\rangle$  possesses a  $r^4$  in the algorithm. We tested that more polynomials (up to  $r^7$ ) can provide negligible conversion efficiency improvement while costing much more time to run the code. The phase distributions on two SLMs are shown in Fig. 4.5(a). The intensity distributions of the input and output fields are shown in Fig. 4.5(b). It can be seen that the converted mode is similar but not identical to  $|0\rangle_p$  due to the non-unity conversion efficiency. The conversion efficiency in our simulation is  $|\langle 0_p \rangle 0'_p|^2 = 82.7\%$  where  $|0'_p \rangle$  denotes the converted mode by SLMs. Due to this non-unity conversion efficiency, the minimum crosstalk of the superposition mode converter can be readily calculated to be 4.5%. In general, a mode conversion requires multi-plane iterations [47], and therefore our implementation to transform  $|1\rangle_p$  to  $|0\rangle_p$  is not sufficient to achieve a unity conversion efficiency. However, it is straightforward to cascade more SLMs to reduce the crosstalk of the superposition mode sorter [143].

The radial superposition mode sorter we experimentally implemented is depicted in Fig. 4.6(a), and its performance is evaluated by measuring the crosstalk matrix. We first remove the radial mode converter such that the set-up becomes a common-path radial mode sorter. We characterize the performance of this commonpath radial mode sorter by measuring output power of two output ports of the PBS and the result is presented in Fig. 4.6(b). The crosstalk of the radial mode sorter is



Figure 4.6: (a) Experimental realization of a common-path radial superposition mode sorter. By removing the radial mode converter in the dashed box it becomes a common-path radial mode sorter. (b) The measured crosstalk matrix of the common-path radial mode sorter. (c) The measured crosstalk matrix of the common-path radial superposition mode sorter.

around 2.6%, which is defined as the power in the wrong port divided by the total output power when a radial mode is injected. We note that this sorter can also be applied to HG modes because both LG modes and HG modes are the eigenmodes of the fractional Fourier transform. We then cascade the radial mode converter and inject  $|p_L\rangle$  and  $|p_R\rangle$  mode to the superposition mode sorter. The measured crosstalk is shown in Fig. 4.6(c), which is around 7.4%. As mentioned above, the crosstalk of the superposition mode sorter can in principle be further decreased by cascading mode SLMs to perform the mode conversion.



Figure 4.7: Schematic of the QKD protocol involving three transverse degrees of freedom. Relay lenses are omitted for simplicity.

#### 4.3 **Results and discussions**

Having constructed the sorters for both mutually unbiased bases, we next demonstrate the implementation of an 8-dimensional QKD protocol involving all three degrees of freedom. A schematic diagram of the setup is presented in Fig. 4.7. A He-Ne laser is modulated by an acousto-optic modulator (AOM) to generate 200 ns optical pulses. The average photon number in each pulse is attenuated by neutral density filters and crossed polarizers to  $\mu = 0.1$ . Computer-generated holograms imparted on SLM are used to generate LG modes of beam waist size  $w_0 = 462.3 \,\mu\text{m}$  at the first diffraction order [34]. A combination of PBS and HWP is used to measure the polarization states in both mutually unbiased bases. Then an OAM mode sorter consisting of an un-wrapper and a phase corrector [121] is used to sort the two OAM modes  $|-2\rangle_{\ell}$  and  $|2\rangle_{\ell}$  [91]. An additional beamsplitter (BS) is needed to recombine the separated modes so as to sort OAM superposition states  $|\ell_D\rangle$  and  $|\ell_A\rangle$ . We note that the sorting mechanism of OAM superposition states is the same as that of the superposition of radial modes. However, the mode converter is not needed because the log-polar transformation induced by the OAM mode sorter already converts the OAM modes to have the same spatial shape, and therefore simply injecting the sorted OAM modes to a beamsplitter (i.e., a two-dimensional quantum F-gate) can efficiently separate the OAM superposition states. The sorting mechanism for the radial degree of freedom follows the scheme presented in Fig. 4.2. After the polarization state is determined, the common-path radial mode sorter is be used to map different radial modes to different polarizations, therefore one can use a PBS to detect the radial quantum number. A radial mode converter needs to be inserted to sort the radial superposition modes  $|p_L\rangle$  and  $|p_R\rangle$  as discussed above. All sorted photons are collected by multi-mode fibers (MMFs) and detected by single-photon avalanche photodiodes (APDs, Perkin Elmer SPCM-AQRH-14-FC). We note that since only four APDs are available at the time of performing experiment, at Bob's side the data is collected for elements in each basis separately and combined later. Since the output of the OAM mode sorter has a small size, we use a 10X beam expander (GBE10-A, Thorlabs) to expand the beam before the MMFs.

To evaluate the performance of the superposition mode sorter, we measure the crosstalk matrix with highly attenuated coherent states and the result is presented in Fig. 4.8. The crosstalk ranges from 6.0% to 16.7%, with an average of 11.7% well below the 8-D QKD error threshold 24.7% [21]. Here the crosstalk is the probability that the photon triggers the wrong APDs when Bob is detecting in



Figure 4.8: Conditional probability matrix measured with  $\mu = 0.1$  pulses. Each element in this matrix represents the probability that Bob detects the corresponding symbol conditioned on the symbol sent by Alice.

the correct basis. To calculate the mutual information, we assume a uniform error rate for detecting each mode by using the averaged crosstalk, and the mutual information can be expressed as [21]

$$I_{\rm AB} = \log_2 d + F \log_2 F + (1 - F) \log_2 \left(\frac{1 - F}{d - 1}\right),\tag{4.9}$$

where d = 8, the average error rate  $\delta = 11.7\%$ , and F is the probability of correct measurement  $F = 1 - \delta = 88.3\%$ . With these numbers we can immediately get  $I_{AB} = 2.15$  bits per sifted photon. We also note that the SLM used in our experiment can be readily replaced by passive, polarization-dependent liquid crystal retarder to realize a scalable, low-cost sorter.

Compared to a QKD protocol with OAM encoding only [19], our protocol employs the slowly divergent LG modes and are thus practical for free-space links with finite-sized apertures [120]. In a realistic free-space channel, atmospheric turbulence can lead to modal crosstalk and reduce the transmission rate. A recent experiment [128] has suggested the potential of LG modes in a free-space channel in an urban environment. The intensity pattern of the transmitted beam can remain recognizable after a 1.6-km free-space link, and thus adaptive optics can be potentially used to mitigate turbulence [128], which can be subject to future study. In a realistic free-space link, in addition to the spatial distortion induced by turbulence which can be corrected by adaptive optics, different LG modes accumulate different amount of Gouy phase which can affect the sorting of superposition modes [131]. The Gouy phase for a LG mode can be written as  $\phi = (2p + |\ell| + 1) \arctan(z/z_R)$ , where  $z_R$  is the Rayleigh range and z is the propagation distance. As proposed in [131], pre-compensation can be used in mode preparation at the transmitter's side to guarantee that the mode-dependent phase is cancelled at the receiver's side. Furthermore, the phase-dependent phase can also be removed at the receiver's side. Here we take the scheme in Fig. 4.2(a) as an example to show how this mode-dependent phase can be removed. Since a radial mode sorter is used in the superposition mode sorter, individual radial modes are separated to different paths. By simply adjusting the path lengths for each radial mode, a mode-dependent phase can be added to cancel the Gouy phase. In our experiment, the radial modes are sorted to horizontal and vertical polarizations respectively, and by adding a constant phase on the subsequent polarizationsensitive SLM, we are able to compensate the relative phase between different radial modes. And the same method can be applied to the OAM index. On the other hand, the large communication bandwidth offered by LG modes [146] can be obtained with less difficulty for a smaller distance and thus provides benefits to short-range optical interconnects [147].

In conclusion, we provide an experimental demonstration of a QKD protocol which encodes information using all possible transverse degrees of freedom, i.e. polarization, radial, and OAM modes, with a resulting transmission of 2.15 bits per sifted photon. A sorting scheme for superposition spatial modes is implemented to enable this 8-D protocol and can find direct application in other fields such as super-resolution imaging and high-dimensional entanglement certification. We believe our demonstration opens up a way to fully exhaust the information resources of finite-sized apertures and therefore reach the capacity limit of a communication channel. The slowly divergent LG beams also make this protocol promising for a free-space communication network.

### **Chapter 5**

# Digital phase conjugation for free-space communication

In recent decades, QKD has attracted increasing interest because it can guarantee communication security based on fundamental laws of quantum mechanics [148]. Free-space QKD [149–156] can guarantee communication security between mobile nodes such as aircraft and satellites. In addition, free space presents lower loss than fibers and thus is favorable to loss-sensitive applications such as quantum teleportation [157] and entanglement distribution [158]. Due to the intrinsically low brightness of quantum light sources, the secure key rate of QKD is significantly lower than the data transfer rate of classical communication protocols. Thus, it remains highly desirable to enhance the secure key rate of QKD. The spatial degree of freedom is a promising candidate for boosting capacity of both quantum and classical communication through MDM [22, 159] or high-dimensional encoding [3, 19, 20, 50–52, 128, 160] and is compatible with polarization- and wavelength-division multiplexing. In particular, slowly diverging spatial modes such as OAM modes are commonly used as a basis set in free-space communication

tion compared to alternative basis sets such as discrete spot arrays [85] which are unsuitable for a long-distance link. However, atmospheric turbulence inevitably leads to strong modal crosstalk between spatial modes [49, 114], which severely degrades the channel capacity of a free-space link. We next summarize several previous works to show the typical level of crosstalk between OAM modes in a turbulent, outdoor free-space channel. In a 150-m link [50], the crosstalk fluctuates between 60% and 80% depending on time. In a 300-m intra-city link [20], the crosstalk is 11% with a mode spacing  $\Delta \ell$  of 4. In a 340-m cross-campus link [51], the crosstalk is measured to be in the range between 70% and 80%. In a 3-km link [52], a camera is used to measure the images of bright OAM superposition modes, and an artificial neural network (ANN) is applied for image recognition, resulting in a bit error rate (BER) of  $1.7 \times 10^{-2}$ . Hence, the turbulence can be a serious concern for crosstalk-sensitive applications such as QKD. A more comprehensive summary is listed in Table 5.1.

Adaptive optics is the most common method for turbulence correction and has been widely adopted for astronomical imaging [53]. A conventional adaptive optics system consists of a wavefront sensor and a deformable mirror at the receiver. The wavefront sensor measures the aberrated phase of an incoming beacon beam (typically a Gaussian beam), and subsequently the deformable mirror corrects the phase aberration of the incoming beam based on the feedback from wavefront sensor as post-turbulence compensation [54]. However, as shown in Fig. 5.1(a), different OAM modes (LG modes with radial index p = 0 and OAM index  $\ell = -1, 0, 1$ ) exhibit mode-dependent amplitude and phase distortions af-

No.	Link length	Parameters	Results	Compensation method	
1	100 m [161]	$\lambda$ =1550 nm,	Crosstalk≈20% with	Fast steering mirror	
1		large $N_f$	a mode spacing $\Delta \ell$ of 2		
2	120 m [159]	$\lambda$ =1550 nm	Crosstalk=4.8% with	None	
L 2		$N_f = 20.8$	a mode spacing $\Delta \ell$ of 2	INOILE	
			Crosstalk≈60%		
3	150 m [50]	$\lambda$ =532 nm	during the daytime and	None	
		$N_f = 97.0$	$\approx 80\%$ during the nighttime	None	
		-	with a mode spacing $\Delta \ell$ of 1		
4	300 m [20]	$\lambda$ =850 nm	Crosstalk=11% with	Post-selecting lucky	
		$N_f = 22.1$	a mode spacing $\Delta \ell$ of 4	beams	
	340 m [51]		Crosstalk can be reduced		
5		$\lambda$ =633 nm	from 80% to 77%	Adaptive optics	
		$N_f = 4.89$	by using compensation	and fast steering mirror	
		-	with a mode spacing $\Delta \ell$ of 1		
6	1.6 km [128]	$\lambda$ =809 nm	Crosstalk≈90%	None	
		$N_f = 8.7$	with a mode spacing $\Delta \ell$ of 1	INOILE	
7	3 km [52]	$\lambda$ =532 nm	Bit error rate= $1.7 \times 10^{-2}$	Artificial neural network	
8	143 km [134]	$\lambda$ =532 nm	Bit error rate= $8.3 \times 10^{-2}$	Artificial neural network	
	240 - [4]		Crosstalk reduced from		
9		) 790	37.0% to 13.2% with		
	540 III [4]	$\lambda = 780 \text{ mm}$	a mode spacing $\Delta \ell$ of 1, and	Phase conjugation	
		$10_{f} = 9.4$	from 10.0% to 3.4% with		
			a mode spacing $\Delta \ell$ of 2		

Table 5.1: A summary of previous works for free-space OAM communications.  $\lambda$ : wavelength.  $N_f$ : Fresnel number product.

ter propagation through the same turbulent link [51]. Therefore, a conventional post-turbulence single-plane phase-only adaptive optics system is unable to correct both amplitude and phase distortions for different OAM modes simultaneously, even in principle [55, 56]. Furthermore, the effectiveness of adaptive optics for OAM communication has mostly been tested in numerical simulations [56–58] or in lab-scale links with emulated, slowly varying, fully controllable turbulence [59–62]. To the best of our knowledge, there is only one experimental demonstration using adaptive optics for OAM communication through an outdoor link



Figure 5.1: (a) Simulation of OAM mode propagation through turbulence. Amplitude and phase distortion of different modes cannot be corrected by post-compensation adaptive optics simultaneously. (b) The spatial modes received by Bob are further distorted if Alice uses a standard mirror to reflect the light. (c) High-fidelity OAM modes can be received by Bob if Alice uses a phase-conjugating mirror to reflect the beams.

[51], and the crosstalk is reduced from 80% to 77% by using both adaptive optics and a fast steering mirror simultaneously. This level of crosstalk is too large to guarantee secure QKD [21]. Therefore, adaptive optics has achieved very limited performance enhancement in outdoor free-space OAM communication links despite numerous simulations and lab-scale experiments. Other methods for turbulence suppression, such as a multiple-input multiple-output (MIMO) algorithm [63] and an ANN [52], cannot be applied to QKD because these algorithms require a large number of photons for digital signal processing and thus are inappropriate for quantum applications that operate at a single-photon level.

#### 5.1 Concept of phase conjugation

Here we propose and demonstrate that digital phase conjugation [5, 162–166] can be used to effectively suppress atmospheric turbulence for OAM communication through a 340-m free-space link. Phase conjugation is also referred to as time reversal, and Fig. 5.1 illustrates how to transmit high-fidelity spatial modes from Alice to Bob by using phase conjugation. As shown in Fig 5.1(a), Bob first transmits standard OAM modes to Alice, and the modes received by Alice are distorted both in amplitude and phase. By contrast, perfect OAM modes can be transmitted to Bob if Alice uses a phase-conjugating mirror as shown in Fig 5.1(b). For an arbitrary incident spatial mode  $A(x, y)e^{i\phi(x,y)}$ , the mode reflected by a phase-conjugating mirror becomes  $A(x, y)e^{-i\phi(x,y)}$ . After propagation through the same turbulent link, the OAM modes received by Bob become the phase conjugate of the originally transmitted OAM modes and can in principle have a perfect spatial mode quality [167], assuming a link without beam clipping. The phase-conjugating mirror can be digitally implemented by a phase-only spatial light modulator (SLM) [162], because spatial amplitude and phase modulation can be simultaneously realized by a diffractive hologram [25]. In order to compensate for time-varying turbulence, the hologram needs to be updated dynamically in real time. From a technical point of view, the spatial modes transmitted from Bob to Alice can be regarded as probe beams that enable fast characterization of turbulence and thus allow Alice to perform pre-turbulence mode generation for each spatial mode. Although digital phase conjugation has been employed for



Figure 5.2: Schematic of the experiment. The upper left image shows the aerial view of the freespace link and is courtesy of Map data ©2020 Google. The retroreflector is installed on a building rooftop. Digital phase conjugation is realized by displaying a computer-generated hologram on SLM 2 based on the feedback from Cam 1. AWG: arbitrary waveform generator. BS: beamsplitter. PBS: polarizing beamsplitter. FC: fiber coupler. Ch. 1: Channel 1. Ch. 2: Channel 2.

aberration correction in biological tissues [162, 163] and MMFs [5, 164–166], it has not been experimentally applied to free-space optical OAM communication.

#### 5.2 Realization of digital phase conjugation

Our phase conjugation experimental schematic is presented in Fig. 5.2. We first characterize the modal crosstalk matrix of the phase conjugation system, and then use the setup to realize a two-channel OAM communication system. In the experiment, Bob uses a 780 nm laser (DL pro, Toptica) as light source and a phase-only SLM (Pluto 2, Holoeye) as SLM 1. A static diffractive hologram is displayed on SLM 1 (Pluto 2, Holoeye) to generate two 780 nm OAM probe beams of horizontal and vertical polarization, respectively. The beam waist radius of the OAM

modes before Bob's telescope is  $w_0 = 0.7$  mm. These two OAM probe beams are combined by a PBS, expanded by a telescope, and then transmitted to Alice through a retroreflector. These two orthogonally polarized probe beams allow simultaneous phase conjugate generation for two different OAM modes, which facilitates the realization of two-channel OAM communication system discussed later. The telescope at Bob's side consists of a f = 35 mm lens (LA1027-B, Thorlabs) and a f = 500 mm lens (LA1380-AB, Thorlabs). The retroreflector (#49-672, Edmund Optics, 127 mm diameter) is installed on a building rooftop that is 170 m away, resulting in a round trip distance of 340 m. The telescope at Alice's side consists of a f = 11 mm lens (C220TMD-B, Thorlabs) and a f = 400 mm lens (LA1725-A, Thorlabs). At Alice's side, a PBS is used to separate the two aberrated OAM modes, and a coherent reference plane wave is combined with two separated OAM probe beams by a beamsplitter (BS). A camera (Cam 1, BFS-U3-04S2M-CS, FLIR) is used to record the interference fringes, and a 45° polarizer is inserted before Cam 1 to enhance the interference pattern visibility. Through the standard off-axis holography analysis [168], the amplitude and phase of the received modes can be retrieved with a single-shot measurement from Cam 1. Hence, Cam 1 is used as a wavefront sensor in our setup. Alice uses Cuda C++ language on a desktop computer (CPU: Intel i7-9700K, GPU: Nvidia RTX 2070 Super) to perform off-axis holography analysis, compute the hologram, and display the hologram on SLM 2 (HSP-1920-488-800, Meadowlark Optics). To provide the coherent reference plane wave for off-axis holography, a single-mode fiber is used to guide the continuous-wave 780 nm light source from



Figure 5.3: Procedure for digital phase conjugation. (a) The interference fringe pattern recorded by Cam 1. (b) The fast Fourier transform (fft) of interference fringe. (c) The first order is selected by a window and shifted to the center. (d) The inverse fast Fourier transform (ifft) gives the amplitude and phase of the received probe beam. (e) The digital phase conjugation is implemented by flipping the sign of the phase. (f) The computer-generated phase-only diffractive hologram that can generate the phase-conjugated OAM mode at the first diffraction order.

Bob to Alice. We emphasize that this single-mode fiber can be avoided by using alternative wavefront sensors such as a Shack-Hartmann sensor [169] or complex field direct measurement [170].

The computational procedure for off-axis holography and digital phase conjugation is presented in Fig. 5.3. One example of the interference fringes recorded by Alice is shown in Fig. 5.3(a). We perform fast Fourier transform of the interference fringe, select the first order and shift it to the image center. By performing an inverse fast Fourier transform, the amplitude and phase of the received probe beam can be obtained. Digital phase conjugation is implemented by flipping the sign of the phase. Then we use the type-2 method in [25] to generate the phaseonly diffractive hologram, which can be used to generate the phase conjugate of the probe beam at the first diffraction order with desired amplitude and phase. Based on the measured amplitude and phase of the aberrated OAM modes, Alice computes the corresponding diffractive hologram and displays it on SLM 2 to generate the phase conjugates of the received OAM modes with desired amplitude and phase. Alice uses a 785 nm laser diode (LP785-SF20, Thorlabs) as the light source and transmits the phase-conjugated modes to Bob. Due to the negligible dispersion of free space, the difference in wavefront distortion between 780 nm and 785 nm wavelength can be ignored [51]. Bob uses Cam 2 (BFS-U3-16S2M-CS, FLIR) to record images of the received modes. In the experiment we use seven LG modes of  $\ell = -3, -2, \cdots, 3$  to test the performance of the phase conjugation system. The aperture diameter of both telescopes is 5 cm, resulting in a Fresnel number product of  $N_f = D^2/\lambda z = 9.4$ , where D = 5 cm,  $\lambda = 780$  nm, and z = 340 m. The beam waist radius of the OAM modes is  $w_0 = 10$  mm after beam expansion of the telescope. The turbulence structure constant [115]  $C_n^2$  is measured to be in the range of  $2.2 \times 10^{-15}$  m<sup>-2/3</sup> to  $8.6 \times 10^{-15}$  m<sup>-2/3</sup>, the Fried parameter  $r_0$  ranges from 0.16 m to 0.07 m, and thus  $D/r_0$  is between 0.31 and 0.70.

Bob uses a separate active area on SLM 2 as the independent SLM 3 to implement spatial mode demultiplexing. Mode-multiplexed holograms [50, 160] are used to project an optical beam onto three different OAM modes simultaneously. We use Cam 3 (BFS-U3-16S2M-CS, FLIR) to measure the diffraction orders at the Fourier plane of SLM 3, and a few experimentally recorded images are shown in Fig. 5.4 for a phase-conjugated  $\ell = -2$  mode. We show the first-order diffractions and other irrelevant diffraction orders are blocked. For each diffraction order, we use a window of  $7 \times 7$  pixel size at the beam center as depicted by the red boxes



Figure 5.4: Experimentally recorded images for crosstalk matrix measurement using modemultiplexed computer-generated holograms. The input beam is a phase-conjugated  $\ell = -2$  mode. The hologram performs OAM projection measurements onto (a)  $\ell = -2, -1, -3$ , (b)  $\ell = -2, 1, 0$ , and (c)  $\ell = -2, 3, 2$ , respectively. The red boxes denote the windows used for crosstalk matrix measurement. Scale bar: 0.5 mm.

in Fig. 5.4 and calculate the total power inside the window. We capture 500 images for each hologram to get the statistical average of the relative power ratio between different modes. For each transmitted mode, three mode-multiplexed holograms are used to measure the crosstalk spectrum for  $\ell = -3, -2, \dots, 3$ . We repeat this procedure for seven OAM modes to get a  $7 \times 7$  crosstalk matrix. The optical signals with on-off-keying modulation are converted to voltage waveforms by using a photodetector (APD130A, Thorlabs) connected to an oscilloscope. For each received voltage waveform, we specify a voltage threshold and convert the signals to binary bit streams by comparing the signals to the threshold. The voltage threshold is tuned independently for each signal stream to reach the minimized the BER.

Due to atmospheric turbulence and aberration of the telescope system, the OAM probe beams received by Cam 1 at Alice's side exhibit clear distortions as shown in the top row of Fig. 5.5(a). Alice generates the phase conjugates of



Figure 5.5: Typical examples of the spatial modes received by Alice (top row) and Bob (bottom row). Scale bar: 0.5 mm.

these aberrated modes and transmits them to Bob. The phase-conjugated modes received by Cam 2 at Bob's side are shown in the bottom row of Fig. 5.5, which exhibit improved mode fidelity compared to those received by Alice. To quantify the modal crosstalk of the phase-conjugated modes, we display a densely encoded diffractive hologram on SLM 3 as a mode demultiplexer [160], and a camera (Cam 3) is placed at the Fourier plane of SLM 3 to measure the crosstalk matrix [50]. When Alice transmits standard OAM modes, the crosstalk matrix of the modes received by Bob is shown in Fig. 5.6(a). The average value of the diagonal elements is 63.0%, and thus the average crosstalk is 37.0%. Therefore, this link cannot support secure QKD with OAM encoding in the absence of phase conjugated OAM modes to Bob, the average crosstalk is reduced to 13.2% as shown in Fig. 5.6(b), which allows for secure QKD operating at a single-photon level. In addition, it is well known that the modal crosstalk can be reduced by in-



Figure 5.6: (a) Experimentally measured crosstalk matrix without phase conjugation. The average crosstalk is 37.0%. (b) Crosstalk matrix with phase conjugation. The average crosstalk is 13.2%.

creasing the mode spacing  $\Delta \ell$  [49]. We calculate the crosstalk matrix with a mode spacing  $\Delta \ell$  of 2 by post-selecting the data of  $\ell = -3, -1, 1, 3$ , and the average crosstalk can be further reduced from 10.0% to 3.4% by using phase conjugation as shown in Fig. 5.7. These results are the lowest crosstalk ever achieved in an outdoor free-space link to the best of our knowledge. It can be seen in Table 5.1 that only our phase conjugation method presents significant modal crosstalk suppression, and our experiment has the lowest modal crosstalk. It should be noted that the works in No. 7 and No. 8 report their results in terms of bit error rate rather than crosstalk. In addition, the artificial neural network used in these works requires data acquisition using a camera. Therefore, the communication speed is fundamentally limited by the camera image acquisition rate, which is on the order of kHz. Furthermore, since high signal-to-noise ratio is needed for image processing, this method cannot be directly used in quantum communications op-



Figure 5.7: (a) Crosstalk matrix using a mode spacing  $\Delta \ell$  of 2 without phase conjugation. The average crosstalk is reduced to 10.0%. (b) Crosstalk matrix using a mode  $\Delta \ell$  spacing of 2 with phase conjugation. The average crosstalk is reduced to 3.4%.

erating at a single-photon level. Although we use classical light to characterize the crosstalk matrix, we emphasize that our method can be readily applied to quantum applications that operate at a single-photon level.

### 5.3 Results and discussions

Ideally, a realization of phase conjugation can completely eliminate the aberration and achieve zero crosstalk. Here we attribute the nonzero crosstalk observed in our experiment to the following reasons. First, the operational bandwidth of our digital phase conjugation system is limited. The image transfer time from Cam 1 to computer memory is 4 ms, the computation time for diffractive hologram generation is 1 ms, and the refresh time of SLM 2 is 5 ms, resulting in a total response time of  $\approx 10$  ms and hence an operational bandwidth of 100 Hz.



Figure 5.8: Eye diagrams of different realizations when transmitting the  $\ell = 2$  and  $\ell = 3$  modes and receiving the  $\ell = 2$  mode. The diagrams with phase conjugation are displayed in the top row, and those without phase conjugation are shown in the bottom row.

By contrast, the characteristic frequency of turbulence can be tens of to hundreds of hertz [53]. We believe that the operational bandwidth can be improved to exceed 1 kHz with faster devices such as a 100 kHz wavefront sensor [169] and 22 kHz SLM [35]. Second, the mode fidelity of our phase conjugate generation is not perfect. The phase conjugate generation fidelity can be further improved by calibrating and correcting the residual aberration of the SLM. Third, beam clipping should be avoided in a free-space link in order to enable a perfect realization of phase conjugation. In our experiment, beam clipping can occur at the telescope aperture as well as at the retroreflector, and the link transmittance for  $\ell = \pm 3$  is  $\approx 10\%$  lower than that of  $\ell = 0$  as shown in Fig. 5.9. It should be



Figure 5.9: The transmittance of the free-space link for different OAM modes. The average transmittance for  $\ell = 0$  is normalized to unity. The error bar represents one standard deviation.

noted that the retroreflector is unnecessary and can be removed in a realistic freespace link. In addition, low-cost and large-diameter Fresnel lenses (1 m diameter lens is commercially available [171]) can be used in a long-distance link to avoid beam clipping. The static aberration of the Fresnel lens should not be a concern because it can be corrected by phase conjugation as part of the overall channel aberration. These limitations impose constraints on the link length in real-world applications. A long free-space link suffers stronger and faster turbulence, and the Fresnel number product is typically smaller for long links due to the limited lens size. Therefore, more technical efforts are needed to enable free-space optical MDM through a long-distance link.

It should be noted that the data transfer rate of OAM communication is not limited by the SLM refresh rate but is decided by the modulation speed of the



Figure 5.10: Time-resolved BER measurement without phase conjugation. The average BER is  $3.6 \times 10^{-3}$  without phase conjugation. The inset shows the eye diagrams acquired by oscilloscope.

transmitter [62]. To clarify this point, we demonstrate a proof-of-principle classical communication system with two-channel OAM multiplexing. We choose  $\ell = 2$  for channel 1 and  $\ell = 3$  for channel 2. Bob generates horizontally polarized  $\ell = 3$  mode and vertically polarized  $\ell = 2$  mode and transmits them to Alice. It should be noted that the polarization has negligible effect on beam propagation due to the small birefringence of turbulent free space. Hence, Alice can measure two aberrated OAM modes simultaneously with a single-shot measurement. We emphasize even though we are using orthogonally polarized OAM modes to facilitate turbulence characterization, this is unnecessary and can be avoided by transmitting different OAM modes in different time slots to Alice using a spatial mode switch [172]. The intensity of the 785 nm laser diode at Alice's side is mod-



Figure 5.11: Time-resolved BER measurement with phase conjugation. We detected 0 bit errors within  $8 \times 10^6$  received bits with phase conjugation, and therefore the average BER is less than  $1.3 \times 10^{-7}$ . The inset shows the eye diagrams acquired by oscilloscope.

ulated at the rate of 20 Mbps with on-off keying format. The modulation rate is solely limited by our modulator bandwidth and can be readily improved to Gbps level by using commercially available high-speed modulators. A fiber coupler is used to split the beam, and a 10 m fiber delay line is used to de-correlate the signal streams. The two beams illuminate separate areas of SLM 2 to generate the corresponding phase conjugate of the two OAM modes transmitted by Bob. The two phase-conjugated OAM modes are combined by BS 1 and then transmitted to Bob. It should be noted that both phase-conjugated modes are vertically polarized, and the horizontal polarization is an unused degree of freedom that can be further adopted for polarization encoding or multiplexing if needed. In the schematic



Figure 5.12: Time-resolved BER measurement without phase conjugation. The average BER is  $2.7 \times 10^{-3}$  without phase conjugation. The inset shows the eye diagrams acquired by oscilloscope.

shown in Fig. 5.2, we use BS 4 at the receiver to split the beam and subsequently use SLM 3 to perform projection onto two OAM modes. In the experiment we omit BS 4 and perform different OAM projections by switching the hologram for simplicity. The projection measurement realized by SLM 3 can be replaced by a low-loss OAM mode sorter [43] for loss-sensitive applications such as QKD. Within every 4 s we collect  $8 \times 10^4$  bits from the oscilloscope, and we collect a total of  $8 \times 10^6$  bits over 400 s. The eye diagrams at different times for  $\ell = 2$  is shown in Fig. 5.8. It can be seen that the data transfer rate is determined by the intensity modulator we use and is not limited by the refresh rate of SLM. Slight power fluctuation can be observed in the eye diagrams between different realizations, and we attribute it to the laser diode power instability as well as the SLM



Figure 5.13: Time-resolved BER measurement with phase conjugation. We detected 0 bit errors within  $8 \times 10^6$  received bits with phase conjugation, and therefore the average BER is less than  $1.3 \times 10^{-7}$ . The inset shows the eye diagrams acquired by oscilloscope.

phase flickering. In addition, we can see that the signal streams with phase conjugation exhibits lower crosstalk than those without phase conjugation. Off-line digital signal processing is performed to analyze the signal, and the average BER is  $3.6 \times 10^{-3}$  without phase conjugation. By contrast, we detected 0 bit errors when phase conjugation is performed, implying a BER lower than  $1.3 \times 10^{-7}$ . The minimum measurable BER is restricted by the memory capacity of the oscilloscope in our experiment. We measure the time-resolved BER, and the BER for  $\ell = 2$  channel without and with phase conjugation are presented in Fig. 5.10 and Fig. 5.11, respectively. In the absence of phase conjugation, the BER can be as high as  $3.6 \times 10^{-3}$ , while it can be reduced to be less than  $1.3 \times 10^{-7}$  in the presence

of phase conjugation. Similar results can be obtained for  $\ell = 3$  channel as shown in Fig. 5.12 and Fig. 5.13. It can be seen that phase conjugation can significantly reduce the BER. Moreover, we believe that GHz and higher modulation speed can be achieved simply by using faster modulators and detectors [22]. Finally, we wish to emphasize that the previously demonstrated lab-scale adaptive optics system with pre-turbulence compensation by Ren et al. [60] is fundamentally different from our scheme for two reasons. First, we transmit different OAM modes as probe beams, allowing for mode-dependent pre-turbulence compensations for individual modes. By contrast, Ren et al. always use a Gaussian mode with  $\ell = 0$ as the probe beam. Second, we use diffractive holograms on an SLM to control both the amplitude and phase of each phase-conjugated mode, while Ren at al. use an SLM to apply a phase-only compensation to all different OAM modes simultaneously. Due to these limitations, the pre-turbulence compensation demonstrated in [60] is essentially equivalent to the conventional post-turbulence compensation with back-propagating beams and thus does not exhibit a better performance than the conventional adaptive optics.

Based on the low crosstalk and high communication bandwidth of the system, here we propose a practical, scalable free-space QKD system with N-channel OAM multiplexing using phase conjugation for turbulence suppression. The schematic of the proposed QKD system is shown in Fig. 5.14. At Bob's side, an SLM is used to generate and switch sequentially among N OAM modes. High-speed OAM mode switching can be readily achieved by using a digital micromirror device at 22 kHz [35] or an acousto-optic modulator at 500 kHz [172]. Alice uses a



Figure 5.14: Proposed free-space QKD system with *N*-channel OAM multiplexing using phase conjugation for turbulence suppression. WFS: wavefront sensor.

wavefront sensor to measure the amplitude and phase of each OAM mode in real time. A densely encoded diffractive hologram [160] can be computed and displayed on an SLM to generate and multiplex *N* phase-conjugated modes simultaneously. The hologram needs to be updated dynamically at a speed faster than the turbulence characteristic frequency as discussed earlier. Although the densely encoded hologram typically has a low diffraction efficiency, this is not a problem for coherent-state-based QKD protocols, because strong loss is inherently needed to attenuate a classical, high-brightness laser to a single-photon level. The standard polarization-encoded decoy-state QKD protocol [152] can be used to enable secure communication, and the secure key rate of each channel is not limited by the SLM refresh rate but determined by the polarization modulation rate which can readily reach GHz level [173]. In fact, by adding a high-speed polarization switch and attenuating the laser diode to a single-photon level, our classical MDM system can be immediately turned to a polarization-encoded OAM-multiplexed

QKD system. Alternative coherent-state-based protocols such as time-bin encoding [174] and continuous-variable encoding [175] are also applicable to our scheme. Furthermore, wavelength-division multiplexing is compatible with our scheme because of the non-dispersive, broadband spectral response of free space. The major advantage of this phase conjugation QKD protocol is the low crosstalk as demonstrated in our experiment, which has not been achieved by any adaptive optics system in an outdoor turbulent link and there is still room for improvement. Moreover, this protocol cannot be replaced by classical turbulence suppression methods such as MIMO and ANN for quantum applications operating at a singlephoton level as discussed earlier.

In conclusion, we experimentally demonstrate turbulence suppression in a 340-m free-space OAM communication link through the use of digital phase conjugation. The crosstalk induced by turbulence can be reduced from 37.0% to 13.2% with phase conjugation, and further down to 3.4% by using a mode spacing  $\Delta \ell$  of 2. We believe that lower crosstalk can be reasonably achieved by using faster equipment in a straightforward manner. A proof-of-principle classical communication system is realized to show the feasibility of high-speed communication with OAM multiplexing. In addition, a practical and scalable scheme for free-space QKD with OAM multiplexing is also proposed and analyzed. Based upon the scalability of the experimental implementation and low crosstalk of the data, we anticipate that digital phase conjugation can be useful to numerous free-space quantum and classical applications that require turbulence suppression.

## **Chapter 6**

## Vectorial phase conjugation for fiber-optic communication

#### 6.1 Concept of vectorial phase conjugation

As discussed in chapter 5, optical phase conjugation [167] is an effective method for modal crosstalk suppression. Nonetheless, optical phase conjugation is investigated mainly for optical imaging [176–183] but only rarely for optical communication until recently [165, 166]. In addition, a perfect realization of phase conjugation should account for not only the spatial degree of freedom but also the polarization. However, all experimental demonstrations in MMFs [164–166, 181– 183] to date have been solely based upon scalar phase conjugation which only accounts for the spatial degree of freedom, while ignoring polarization. As we will show later, the polarization mixing is severe in a long fiber and thus scalar phase conjugation can only succeed in a short MMF ( $\approx 1$  m) [164, 165, 181–183] or a few-mode fiber [166]. It should be noted that while many demonstrations of transfer matrix inversion have taken into account the polarization [72–76], this

j	0	1	2	3	4	5
N	0	1	1	2	2	2
p	0	0	0	0	1	0
$\ell$	0	-1	1	-2	0	2
m	0	0	1	0	1	2
n	0	1	0	2	1	0

Table 6.1: Examples of relation between the single mode index j, mode group number N, LG mode indices  $(p, \ell)$ , and HG mode indices (m, n).

method cannot be used in a long, unstabilized fiber as discussed in chapter 1.

Since a MMF typically supports up to hundreds of modes, here we introduce a single mode index  $j = 0, 1, 2, \cdots$  to simplify the notation of the LG and HG modes for later use. Here we refer to the convention of Zernike polynomials [184] and adopt the following definitions. For a specific mode index j, the mode group number N can be calculated as  $N = \operatorname{ceil}((-3 + \sqrt{9 + 8j})/2)$ . Then we have  $\ell = 2j - N(N+2)$  and  $p = (N - |\ell|)/2$  for the LG mode. Given the one-to-one correspondence between the HG mode and the LG mode [104], we can define  $m = p + \max(\ell, 0)$  and  $n = p - \min(\ell, 0)$  for the HG mode. On the other hand, for a given  $(p,\ell)$  or (m,n), we can calculate j as  $j = [N(N+2) + \ell]/2$  with  $N = 2p + |\ell|$  for LG modes and j = [N(N+2) + m - n]/2 with N = m + n for HG modes. A few examples of conversion relation between j, N,  $(p,\ell)$  and (m,n)are given in Table. 6.1.

Figure 6.1 presents the the conceptual schematic of our experiment. Bob first transmits a probe beam of interest to Alice. Alice uses a polarizing beam displacer to separate the horizontal and vertical polarizations of the scrambled probe beam, and subsequently performs vectorial off-axis holography to measure the ampli-



Figure 6.1: Conceptual illustration of vectorial phase conjugation. Bob transmits a probe beam (denoted by red beams) to Alice. Alice then performs vectorial off-axis holography on her scrambled probe beam and digitally generates the phase conjugate of the scrambled probe beam as the signal beam (denoted by blue beams).  $|H\rangle$  and  $|V\rangle$  stand for the horizontal and vertical polarization state, respectively. PBD: polarizing beam displacer. BS: beamsplitter. HWP: half-wave plate. MMF: multimode fiber.

tude, phase, and polarization of the scrambled probe beam via a single-shot measurement. Hence, Alice can imprint a corresponding computer-generated hologram on her SLM to generate the signal beams, which are the phase conjugate of the displaced, scrambled probe beams. The back-propagating signal beams are transmitted to Bob through the same MMF. Bob also performs vectorial off-axis holography to characterize the unscrambled signal beam. Compared to the scalar phase conjugation presented in chapter 5, the major difference of vectorial phase conjugation is that the polarization of light needs to be taken into account for phase conjugation. This is because the polarization mixing is negligible in free space but can be strong in MMFs.

### 6.2 Implementation of vectorial phase conjugation

We next experimentally demonstrate that digital vectorial phase conjugation can be applied to transmit 210 high-fidelity spatial modes (up to mode group 13) through a 1-km-long, standard, graded-index MMF with the number of used modes limited by the active area size of our SLM. The detailed experimental setup is shown in Fig. 6.2. A 780 nm laser (DL pro, Toptica) is used as the light source, and the light is spatially filtered by a 10-m-long SMF and then collimated to illuminate the SLM. A single SLM (Pluto 2 VIS-020, Holoeye) is used to generate both the probe beam for Bob and the signal beam for Alice with the choice being made by switching the overall phase grating written onto the SLM. A binary phase grating is used to generate complex-amplitude spatial modes in the first diffraction order [34]. We choose the commonly used LG and HG modes for demonstration because they are the eigenmodes of a graded-index MMF [185], which exhibit better robustness and minimized loss during propagation [66] compared to other basis sets and thus enable phase conjugation to a full extent. It has also been shown that secure QKD can be implemented by using HG modes and LG modes as mutually partially unbiased bases [118]. In the experiment, Bob prepares a spatial mode of interest (i.e., a probe beam) such as a standard LG or HG mode and transmits


Figure 6.2: The full schematic of the experimental setup. The same SLM is used to generate the probe beam for Bob (denoted by red lines) and the signal beam for Alice (denoted by blue lines) by switching the phase grating. The multimode fiber spool is shown at the upper right corner and is resting on the optical table without any specialized thermal or mechanical stabilization. The optical table used in the experiment is not floated. The customized PBD is made of a PBS and two mirrors as illustrated by the inset. HWP: half-wave plate. PBS: polarizing beamsplitter. SMF: single-mode fiber. SLM: spatial light modulator. BS: beamsplitter. PBD: polarizing beam displacer. MMF: multimode fiber.

it to Alice through a 1-km-long, standard, graded-index MMF (Clearcurve OM3, Corning). The fiber is comprised of two 500-m-long bare fibers that are spliced together and are free of any specialized thermal or mechanical isolation. The fiber has a core diameter of 50  $\mu$ m and NA = 0.2, therefore supporting  $\approx 400$  modes per polarization at 780 nm.

Bob uses a polarizing beam displacer (MBDA10, Karl Lambrecht) to generate a horizontally polarized probe beam, and the polarization can be adjusted by a subsequent HWP. The generated probe beam is then coupled into a 1-km-long MMF by an aspheric lens L1 (C110TMD-B, Thorlabs). The spatial mode beam waist size in the MMF used in our experiment is  $w_0 = 5.06 \ \mu m$  [116]. After transmission through the fiber, the probe beam received by Alice has a scrambled spatial and polarization profile. The scrambled probe beam is collimated by an aspheric lens L2 (C230TMD-B, Thorlabs), and a subsequent Sagnac interferometer is used as a customized polarizing beam displacer to coherently separate the horizontal and vertical polarization components of the scrambled probe beam to two beams that propagate along the same direction but are transversely displaced with respect to each other [186]. This customized polarizing beam displacer provides more flexibility than the commercially available polarizing beam displacer because the transverse separation between the two displaced beams can be tuned by adjusting the position of mirrors in the Sagnac interferometer. Alice then performs vectorial off-axis holography to measure the spatial and polarization profile of the scrambled probe beams. These two beams are then combined with a coherent, 45° polarized reference plane wave at a beamsplitter, and the resultant interference pattern is recorded by a camera (Camera 1, BFS-U3-16S2M-CS, FLIR). A 1-km-long SMF is used to provide a coherent reference light source to interfere with the scramble probe beams. It should be noted that the SMF providing coherent reference light can be avoided by using a commercial Shack-Hartmann wavefront sensor [169] or vectorial complex field direct measurement [187]. Through off-axis holography [168], the amplitude, phase, and polarization of the scrambled probe beam can be simultaneously determined via a single-shot measurement [187]. Alice then uses a SLM to generate the back-propagating signal beams, which are the phase conjugate of the two displaced, scrambled probe beams. The two back-propagating signal beams are combined coherently by the same polarizing beam displacer. After passing through the same MMF, the signal beam has an unscrambled profile and becomes the phase conjugate of the probe beam originally transmitted by Bob. Vectorial off-axis holography is then performed by Bob to quantitatively characterize the spatial and polarization profile of the unscrambled signal beam. Additional details about the detection and generation of vector beam can be found in [187], and the alignment procedure for phase conjugation can be found in [188]. Bob measures the unscrambled signal beam by another camera (Camera 2, BFS-U3-31S4M-C, FLIR) and performs the digital spatial mode decomposition to obtain the crosstalk matrix.

In our experiment, both Alice and Bob perform data processing for off-axis holography in MATLAB on a desktop computer (CPU: Intel i7-9700K, GPU: Nvidia RTX 2070 Super). The digital data processing for off-axis holography involves several fast Fourier transforms (FFTs), which can be significantly sped up by using a dedicated digital signal processor. The procedure of vectorial off-axis holography is shown in Fig. 6.3. The horizontally polarized component and the vertically polarized component interfere with a 45° polarized plane wave, and the interference pattern is recorded by a camera. Then a Fourier transform is performed, and the first-order component in the Fourier domain is selected and shifted to the center. Finally an inverse Fourier transform is performed, and thus the amplitude and phase of the vectorial speckle pattern are obtained. Ideally, the digital data processing for Alice can even be avoided if the camera for off-



Figure 6.3: Procedure for vectorial off-axis holography. The horizontally polarized beam and the vertically polarized beam are displaced by a polarizing beam displacer and then interfered with a 45° polarized beam. The interference pattern is recorded by a camera. A Fourier transform is performed and the first-order component is selected in the Fourier domain and shifted to the center. Then an inverse Fourier transform is performed to retrieve the polarization, amplitude, and phase of the scrambled probe beam.

axis holography and the SLM for generating phase-conjugated speckles are exactly placed at positions that are imaging planes with respect to each other [188]. When the positions of camera and SLM are perfectly aligned, we can directly imprint the digital interference pattern recorded by the camera onto the SLM without performing any digital signal processing. Instead of carefully aligning the position of the camera, Alice digitally compensates the misalignments such as tip and tilt, transverse and axial displacement, and defocus (see [188]). At Bob's side, we are using a second digital off-axis holography to measure the full received field, which allows us to obtain the crosstalk matrix with reduced experimental com-



Figure 6.4: The measured amplitude, phase, and polarization of the vectorial speckle pattern and phase-conjugated mode for horizontally polarized (a)  $LG_{32}$  and (b)  $HG_{44}$  mode respectively.

plexity. It should be noted that the off-axis holography can be readily replaced by a spatial mode sorter [1, 2, 189] to analyze the crosstalk matrix and enable high-speed spatial mode detection.



Figure 6.5: The unnormalized mode fidelity for phase-conjugated (a) LG modes and (b) HG modes. A single index is used to denote the two mode indices for simplicity, and the light and dark blue bands denote the odd and even mode group number respectively.

## 6.3 Characterization of spatial mode fidelity

### 6.3.1 Crosstalk matrix measurement

Figure 6.4(a,b) shows two examples of experimentally measured scrambled probe beams received by Alice and the unscrambled signal beams received by Bob for horizontally polarized LG<sub>32</sub> and HG<sub>44</sub> modes. For each unscrambled signal beam received by Bob, we digitally project the mode to an orthonormal spatial mode basis set to calculate the crosstalk matrix. We measure the crosstalk matrix for 105 LG modes with  $2p + |\ell| \le 13$  in both horizontal and vertical polarization ba-



Figure 6.6: The mode fidelity and crosstalk distributions for (a) LG modes and (b) HG modes.

sis sets, resulting in a 210×210 crosstalk matrix. The same measurement is also performed for HG modes with  $m + n \leq 13$ . The unnormalized mode fidelity for individual spatial modes (i.e., the diagonal elements of crosstalk matrix) is shown in Fig. 6.5(a) with an average of 85.6% for LG modes and Fig. 6.5(b) with an average of 82.6% for HG modes. Here the crosstalk matrix element is calculated as  $M_{k',k} = |\langle \phi_{k'}^{\text{ideal}} | \phi_{k}^{\exp} \rangle|^2$  and the unnormalized mode fidelity is  $F_k = M_{k,k}$ , where  $|\phi_{k'}^{\text{ideal}}\rangle$  is the ideal spatial mode with mode index k',  $|\phi_{k}^{\exp}\rangle$  is the experimentally measured unscrambled signal beam with mode index k,  $\langle \phi_{k'}^{\text{ideal}} | \phi_{k'}^{\text{ideal}} \rangle = 1$  and  $\langle \phi_{k}^{\exp} | \phi_{k}^{\exp} \rangle = 1$ . The normalized mode fidelity within the 210-mode subspace has an average of 91.5% for LG modes and 89.3% for HG modes, where the normal-



Figure 6.7: Unnormalized  $210 \times 210$  crosstalk matrix on a logarithmic scale for vectorial speckle pattern received by Alice when Bob transmits LG modes in the absence of vectorial phase conjugation.

ized mode fidelity is defined as  $F_k^{\text{norm}} = M_{k,k} / \sum_{k'=0}^{209} M_{k',k}$ .

To aid readers for analyzing the crosstalk, we also present the crosstalk distributions in the following four categories. Here we assume the mode of interest is a horizontally polarized LG mode  $|LG_j, H\rangle$  and the received unscrambled signal beam is  $|\phi\rangle = |\psi_1, H\rangle + |\psi_2, V\rangle$  as an example. The four crosstalk categories are (1) crosstalk from coupling to modes inside the crosstalk matrix with degenerate polarization, which can be expressed as  $C_1 = \sum_k |\langle \phi | LG_k, H \rangle|^2$  for  $0 \le k \le 104$  and  $k \ne j$ . (2) crosstalk from coupling to modes inside the crosstalk matrix with



Figure 6.8: Unnormalized  $210 \times 210$  crosstalk matrix on a logarithmic scale for vectorial speckle pattern received by Alice when Bob transmits HG modes in the absence of vectorial phase conjugation.

orthogonal polarization, which can be expressed as  $C_2 = \sum_k |\langle \phi | LG_k, V \rangle|^2$  for  $0 \le k \le 104$ . (3) crosstalk from coupling to modes outside the crosstalk matrix with degenerate polarization, which can be expressed as  $C_3 = \sum_k |\langle \phi | LG_k, H \rangle|^2$  for  $k \ge 105$  or equivalently  $C_3 = |\langle \psi_1 | \psi_1 \rangle|^2 - C_1 - |\langle \psi_1 | LG_j \rangle|^2$ . (4) crosstalk from coupling to modes outside the crosstalk matrix with orthogonal polarization, which can be expressed as  $C_4 = \sum_k |\langle \phi | LG_k, V \rangle|^2$  for  $k \ge 105$  or equivalently  $C_4 = |\langle \psi_2 | \psi_2 \rangle|^2 - C_2$ . These results are shown in Fig. 6.6.

The 210×210 unnormalized crosstalk matrix for vectorial speckle patterns re-



Figure 6.9: Unnormalized  $210 \times 210$  crosstalk matrix on a logarithmic scale for LG modes received by Bob when performing vectorial phase conjugation.

ceived by Alice when Bob transmits standard LG and HG modes (i.e. in the absence of vectorial phase conjugation) are shown in Fig. 6.7 and Fig. 6.8. Due to the strong spatial mode scrambling, the average unnormalized mode fidelity in this case is  $\approx 1\%$  for both LG modes and HG modes. The  $210 \times 210$  unnormalized crosstalk matrix in the presence of vectorial phase conjugation are presented in Fig. 6.9 and Fig. 6.10. The crosstalk matrix is calculated as follows. The received vectorial mode is denoted as  $|\phi\rangle = |\psi_1, H\rangle + |\psi_2, V\rangle$ , where H and V represent the horizontal and vertical polarization state,  $\psi_1$  and  $\psi_2$  represent the corresponding spatial mode, and  $|\phi\rangle$  is normalized such that  $\langle \phi | \phi \rangle = 1$ . Each element in



Figure 6.10: Unnormalized  $210 \times 210$  crosstalk matrix on a logarithmic scale for HG modes received by Bob when performing vectorial phase conjugation.

the crosstalk matrix is the squared inner product between the received mode  $|\phi\rangle$ and a particular LG or HG mode. As an example, for a horizontally polarized LG mode  $|LG_j, H\rangle$ , the squared inner product can be expressed as  $|\langle \phi | LG_j, H\rangle|^2$ , where  $0 \le j \le 104$  is the single mode index. For a LG mode, it is normalized such that  $\langle LG_j, H | LG_j, H\rangle = 1$  and  $\langle LG_j, V | LG_j, V\rangle = 1$ . Similar normalization is also applied to HG modes. It should be noted that since the spatial modes with  $0 \le j \le 104$  do not form a complete basis set, the sum of each row in the crosstalk matrix is less than unity, i.e.  $\sum_{j=0}^{104} |\langle \phi | LG_j, H\rangle|^2 + |\langle \phi | LG_j, V\rangle|^2 < 1$ . After normalizing the sum of each row of the crosstalk matrix to unity, the mode



Figure 6.11: Normalized mode fidelity for (a) LG and (b) HG modes. The normalized mode fidelity is calculated by dividing the corresponding diagonal element (i.e. the unnormalized mode fidelity) of the crosstalk matrices (shown in Fig. 6.9 for LG modes and Fig. 6.10 for HG modes) by the sum of elements in each column.

fidelity can be higher as shown in Fig. 6.11. It can be seen that the average of normalized mode fidelity is 91.5% for LG modes and 89.3% for HG modes. This is because the crosstalk due to coupling to higher-order modes ( $j \ge 105$ ) is discarded. This is permissible in an experiment because the higher-order modes can in principle be separated by a mode sorter in practical applications and thus does not contribute to the crosstalk.

Although the performance is characterized using a classical light source, our method is readily applicable to QKD by simply attenuating the light intensity to a

single-photon level [190]. Based on the measured crosstalk matrices, the mutual information between Alice and Bob in QKD can be calculated as [21]

$$I_{\rm AB} = \log_2 d + F \log_2 F + (1 - F) \log_2 \left(\frac{1 - F}{d - 1}\right).$$
(6.1)

For LG modes, we have d = 210, F = 91.5% and thus  $I_{AB} = 13.8$  bits per sifted photon. For HG modes, we have F = 89.3% and thus  $I_{AB} = 13.4$  bits per sifted photon. Hence, we can achieve a channel capacity of 13.8 bits per sifted photon with LG modes and 13.4 bits per sifted photon with HG modes for high-dimensional QKD with spatial mode encoding.

### 6.3.2 Experimental generation fidelity of SLM

Here we show that the average mode fidelity without normalization is 85.6% for LG modes and 82.6% for HG modes. The main reason for imperfect mode fidelity is attributed to the imperfect spatial mode generation fidelity of the SLM. To test this hypothesis, we experimentally characterize the fidelity of the probe beam generated by Bob and that of the signal beam by Alice. The product of these two fidelities is referred to as experimental generation fidelity, which is presented as solid lines in Fig. 6.5 for individual spatial modes. In the experiment, Alice uses a SLM to generate the signal beams, and Bob uses the same SLM to generate the probe beams. We characterize the fidelity of the generated probe beams and horizontal polarization component of signal beams using off-axis holography, with the results shown in Fig. 6.12. We take the product of the probe beam fidelity and



Figure 6.12: (a-d) The mode generation fidelity for HG(2,5) and LG(1,4) modes. The HG(2,5) and LG(1,4) modes generated by SLM are shown at the right panel of (a) and (v), and the corresponding ideal modes are shown at the left panel for comparison. The corresponding phase conjugates of the scrambled probe beams measured by off-axis holography are shown at the left panel of (b) and (d), and the generated signal beams are shown at the right panel. The calculated fidelity (F) is listed below the generated modes.

the signal beam fidelity as the experimental generation fidelity, which is a simple estimate of the fidelity of unscrambled signal beams. In Fig. 6.12(a-d) we show the ideal and generated probe beam and signal beam for HG(2,5) and LG(1,4) modes. In general, the mode fidelity for LG modes is slightly higher than that of HG modes, and the LG modes with  $\ell \neq 0$  mode have a higher fidelity than those with  $\ell = 0$ . In addition, the signal beam generation has a slightly lower fidelity than the well-defined LG and HG modes. The maximum experimental generation fidelity we measured is 90.8% for LG modes and 88.4% for HG modes, and the



Figure 6.13: The measured mode fidelity for (a) LG and (b) HG modes received by Bob when Alice performs scalar phase conjugation (i.e. using one polarization only).

average experimental generation fidelity is 85.8% for LG modes and 83.7% for HG modes. It can be seen that the fidelity of unscrambled signal beams is close to the experimental generation fidelity, and thus we believe that the fidelity of unscrambled signal beams can be further increased by using a well-calibrated SLM. Nonetheless, there exists a deviation between phase-conjugated mode fidelity and experimental generation fidelity for high-order modes, and we attribute this difference to the fact that high-order modes are susceptible to mode-dependent loss induced by fiber bending and splicing.



Figure 6.14: The measured vectorial speckle pattern, phase-conjugated spatial mode, and the corresponding intensity for (a)  $LG_{10}$  and (b)  $LG_{32}$ , respectively.

### 6.3.3 Limitations of scalar phase conjugation

We also emphasize that the high mode fidelity is exclusively enabled by vectorial phase conjugation, while scalar phase conjugation can only achieve an average unnormalized mode fidelity of 41.2% for LG modes and 39.7% for HG modes as shown in Fig. 6.13. Here we show the results for scalar phase conjugation. Bob transmits horizontally polarized spatial modes to Alice. While Alice can measure both the horizontal and vertical polarization components of the fiber speckle pattern, she only generates the phase conjugation of the horizontal polarization and ignores the vertical polarization. The unscrambled images via scalar phase conjugation are shown in Fig. 6.14. It can be seen that the mode fidelity of scalar phase conjugation is significantly worse than that of vectorial phase conjugation.



Figure 6.15: The normalized crosstalk matrix in the polarization subspace for horizontal (H), vertical (V), diagonal (D), and anti-diagonal (A) polarizations for  $LG_{02}$ ,  $LG_{12}$ ,  $LG_{22}$ , and  $LG_{32}$  modes.

The average mode fidelity is 41.2% for LG modes and 39.7% for HG modes. The results here confirm the necessity of vectorial phase conjugation.

### 6.3.4 Polarization crosstalk matrix

To evaluate the performance of our system for polarization-based QKD, we measure the  $4 \times 4$  polarization crosstalk matrix for each mode within the corresponding spatial mode subspace. The resultant normalized polarization crosstalk matrices for LG<sub>02</sub>, LG<sub>12</sub>, LG<sub>22</sub>, and LG<sub>32</sub> are shown in Fig. 6.15. The average polarization crosstalk is 0.04% for LG modes and 0.05% for HG modes, which suggests that both the spatial mode and polarization scrambling can be well suppressed through vectorial phase conjugation. These high-fidelity results directly indicate that the polarization-based QKD protocol can be performed through MMFs, and the secure key rate can be significantly boosted by MDM. Furthermore, because these high-fidelity results are obtained in a spliced fiber, we expect that the vectorial phase conjugation would also be realized in a much longer fiber. Here we continue to use LG<sub>j</sub> to show how the calculation is performed. For the received phase-conjugated mode  $|\phi\rangle$ , we first calculate the unnormalized squared inner product for horizontally (H), vertically (V), diagonally (D), and anti-diagonally (A) polarized LG<sub>j</sub> mode as  $F_H^u = |\langle \phi | \text{LG}_j, \text{H} \rangle|^2$ ,  $F_V^u = |\langle \phi | \text{LG}_j, \text{V} \rangle|^2$ ,  $F_D^u = |\langle \phi | \text{LG}_j, \text{D} \rangle|^2$ , and  $F_A^u = |\langle \phi | \text{LG}_j, \text{A} \rangle|^2$ , where  $|\text{D} \rangle = (|\text{H} \rangle + |\text{V} \rangle)/\sqrt{2}$  and  $|\text{A} \rangle = (|\text{H} \rangle - |\text{V} \rangle)/\sqrt{2}$ . Then the normalized fidelity can be calculated as  $F_H = F_H^u/(F_H^u + F_V^u)$ ,  $F_V = F_V^u/(F_H^u + F_V^u)$ ,  $F_D = F_D^u/(F_D^u + F_A^u)$ , and  $F_A = F_A^u/(F_D^u + F_A^u)$ . These four numbers form one row of the crosstalk matrix, and the rest of the matrix can be similarly calculated. The normalized polarization crosstalk matrix for HG modes is also calculated in this way. Since the calculation is performed for a 4×4-dimensional state space, the crosstalk is small. It should be noted that such calculation is permissible because in an experiment one can use a polarization-independent spatial mode sorter and a polarizing beamsplitter to access the 4×4-dimensional state space, and polarization crosstalk that couples to other spatial modes can be discarded.

### 6.3.5 Stability test for vectorial phase conjugation

To overcome environmental instability, which is an inevitable concern for long fibers in a real-world environment, vectorial off-axis holography needs to be performed repeatedly in real time, and the phase pattern on Alice's SLM should be updated accordingly to compensate for instability. In the following, we evaluate the response time of our vectorial phase conjugation system. To perform off-axis holography, we need to retrieve a single-shot image from the camera (which takes



Figure 6.16: Stability test for vectorial phase conjugation. The unnormalized mode fidelity is measured as a function of time. The shaded area corresponds to mode fidelity between 80% and 90%. The solid lines represent the results when the SLM is under active control and the dashed lines represent the results without active control.

2.6 ms) and execute fast Fourier transforms and interpolations as digital data processing (which takes 18 ms on a desktop computer). It should be noted that the data processing time can be significantly reduced by using a dedicated digital signal processor or even eliminated by careful experimental design and alignment as earlier. The response time of our system is therefore only constrained by the refresh rate of SLM, which is 4 Hz in our experiment. However, this constraint can be readily removed by using a commercially available fast digital micromirror device (above 10 kHz refresh rate [191]) or a high-speed SLM (sub kHz refresh rate [32]). In addition, we emphasize that the data transfer rate using each spatial mode is not limited by the refresh rate of SLM or the response time of the timereversal system [166], and the system response time only needs to be faster than the environmental fluctuation rate in order to overcome instability. In this proofof-principle experiment, we implement digital vectorial phase conjugation for one

spatial mode at a time, but we emphasize that our method can be readily used to enable MDM [166]: by separately pre-shaping individual wavefronts of multiple high-speed-modulated signal beams, high-fidelity spatial modes can be recovered at the receiver, and thus the data streams can be demultiplexed with low crosstalk. To test the operation stability of our system, we also measure the unnormalized mode fidelity as a function of time while the SLM is actively updated every  $\approx 30$ seconds for each mode, which is depicted by the solid lines in Fig. 6.16; here the dashed lines represent the mode fidelity in the absence of active control of the SLM. The autocorrelation  $R(\Delta t) = \langle |\langle \phi(t) | \phi(t + \Delta t) \rangle |^2 \rangle$  is calculated according to these data, where  $|\phi(t)\rangle$  is the phase-conjugated mode at time t,  $\langle \cdot \rangle$  is the time average, and  $|\phi(t)\rangle$  is normalized such that  $\langle \phi(t)|\phi(t)\rangle = 1$ . The time for  $R(\Delta t)$  to drop to 1/e is approximately 120 s for LG(0,2) and 100 s for LG(3,2). These results clearly show that our system is able to overcome environmental instability even though the 1-km-long bare fiber is placed on an optical table that has not been floated and is free of any thermal or mechanical isolation. We believe that by using a fast SLM, real-time crosstalk suppression can be achieved even in a harsh environment through a much longer fiber.

## 6.4 Proposed spatial-mode-multiplexed communication protocol

Figure 6.17 presents a practical QKD protocol with MDM. Each spatial mode can be used as an independent channel, with time-bin encoding [174, 192, 193] or



Figure 6.17: Proposed spatial-mode-multiplexed QKD protocol. A densely encoded computergenerated hologram imprinted on a single SLM can be used to simultaneously generate and multiplex a large number of spatial modes.

continuous-variable encoding [194–196] used to guarantee communication security within each channel. In particular, it has been previously demonstrated that a single SLM can be used to simultaneously generate and multiplex up to 105 spatial modes by using a densely encoded computer-generated hologram [160]. To implement the N-spatial-mode-multiplexed QKD protocol, Bob first sequentially transmits N spatial modes of interest to Alice through the fiber. Alice measures the corresponding vectorial speckles using a wavefront sensor (WFS), computes the densely encoded hologram, and then imprints the hologram onto her SLM. The WFS can be realized by either the off-axis holography presented in this work or alternative methods such as the commercial Shack-Hartmann WFS. Alice then prepares N attenuated laser sources with high-speed time-bin encoding or continuous-variable encoding and illuminates the SLM with these N beams incident at different angles. The horizontally polarized light and the vertically polarized light are split by a polarizing beamsplitter (PBS) and are incident at two separation locations on the SLM, which allows for generation of vectorial phase conjugation. Alice's SLM converts each of the N incident beams into the phase conjugation of its corresponding measured vectorial speckle pattern, in addition to multiplexing all the modes to propagate in the same direction. The horizontal and vertical polarization components are recombined by another PBS and finally transmitted to Bob through the MMF. As a consequence, all channels can have a high-fidelity spatial profile at Bob's side, and Bob can use the well-developed LG or HG mode sorter [1, 2, 189] to demultiplex the signals. To overcome environmental instability, Bob needs to periodically send spatial modes of interest to Alice, who updates the phase pattern on the SLM accordingly. The polarization degree of freedom can also be included to further increase the channel capacity. It should be noted that the signal transfer speed is determined by the QKD encoder, not the SLM refresh rate. We emphasize that an analogous protocol, to the best of our knowledge, cannot be realized in a straightforward manner by any alternative methods. Since a complete knowledge of the complex-valued transfer matrix is not needed, our method can be applied to unstabilized, long MMFs outside the laboratory, which is not possible by slow, conventional transfer matrix inversion. MIMO is not applicable to QKD because it requires a large number of photons for digital signal processing. Mode-group excitation only allows for a small number  $(\approx 10)$  of mode groups and is thus unable to fully utilize the channel capacity of the link. Thus, vectorial phase conjugation offers a uniquely practical approach towards spatial-mode-multiplexed quantum communication over realistic, unstable links.

### 6.5 Summary

In summary, we have demonstrated that, through the use of vectorial phase conjugation, we can establish a high-fidelity, 1-km-long communication link that supports 210 spatial modes of a standard MMF. Both spatial mode crosstalk and polarization scrambling in MMF can be well suppressed, which demonstrates the possibility of boosting the communication rate of both classical communication and QKD by either MDM or high-dimensional encoding. In particular, we propose a spatial-mode-multiplexed QKD protocol and show how our method can be used to boost the channel capacity in a straightforward manner. Given the scalability of the experimental implementation and high fidelity of the data, our technique presents a practical approach to a multitude of long-distance quantum and classical applications ranging from MDM to entanglement distribution [70, 71].

### Chapter 7

# **Conclusion and future work**

In this thesis, we have presented an interferometric method to build a radial mode sorter in chapter 2. We next employ the polarization sensitivity of SLM to develop a common-path sorter to enhance the stability, and we have successfully constructed a radial mode sorter as well as a HG mode sorter in chapter 3. We further propose and present how to sort the superpositions of LG modes, and consequently we demonstrate an 8-D QKD protocol using all transverse degrees of freedom in chapter 4. However, the secure key rate in our demonstration is severely limited by the spatial mode switching rate. Here we propose a high-dimensional, high-speed spatial mode switch as shown in Fig. 7.1. The commercial intensity modulators can be synchronized to produce one pulse at a time and thus realizes switching. The SLMs can display a static hologram to generate a spatial mode for each channel. Although this spatial mode switch has a low efficiency, this is not a problem to QKD since strong loss is needed to attenuate a classical laser to a single-photon level. We leave the realization of this high-speed spatial mode switch to future study.



Figure 7.1: Proposed high-speed spatial mode switch. Mod: intensity modulator. SLM: spatial light modulator.

In chapter 5 we show that digital phase conjugation can effectively mitigate the modal crosstalk induced by atmospheric turbulence. A classical MDM system is also reported as a proof-of-principle demonstration, and the bit error rate is reduced from  $3.6 \times 10^{-3}$  to be less than  $1.3 \times 10^{-7}$  through the use of phase conjugation. In chapter 6 we take into account the polarization degree of freedom in phase conjugation and develop the concept of digital vectorial phase conjugation. High mode fidelity is achieved for a large number of spatial modes propagating through a 1-km-long MMF via vectorial phase conjugation. In Fig. 5.14 we propose a practical and scalable scheme for high-speed, spatial-mode-multiplexed QKD through a turbulent link, and a similar scheme for a MMF link is proposed in Fig. 6.17. We leave these proposals to future study.

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