Journal of Nonlinear Optical Physics and Materials
Vol. 5, No. 2 (1996) 387–408

THREE-DIMENSIONAL SIMULATIONS OF STIMULATED BRILLOUIN SCATTERING WITH FOCUSED GAUSSIAN BEAMS

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Received 7 November 1995

We report on a detailed investigation of the process of SBS with focused Gaussian beams via a three-dimensional simulation of the SBS process that includes spontaneous noise. We show that spontaneous noise present at points other than where the initiation of SBS occurs does not significantly affect the Stokes wave. The effects on the Stokes beam of modes not present in the pump wave (nonconjugate modes) are investigated and it is found that the phase of a given Stokes mode is affected by the presence of other modes, a process termed *phase pulling*. We also show that for a focused Gaussian beam the Brillouin gain parameter does not increase linearly with increased pump power once SBS threshold is reached; rather, above threshold the SBS process resembles a reverse saturable absorber for the pump wave at any point between the entrance to the medium and the focal point of the pump beam.

1. Introduction

The process of stimulated Brillouin scattering (SBS) has been investigated for many years and the equations describing the process of SBS are well known[1–3]; however, the nature of the equations is such that analytical solutions have been found only for very narrowly defined situations. Specifically, the case of the undepleted pump has been extensively investigated.

Our purpose here is to investigate the process of SBS and optical phase conjugation in the geometry in which it is usually encountered in the laboratory. That is, we will investigate SBS of focused Gaussian beams with pump powers well above SBS threshold. To accomplish this investigation we will utilize the technique of modal decomposition.

The use of a modal decomposition to study the SBS process has, of course, been used by many researchers.[4–11] In our investigation however, we do not assume an undepleted pump as has often been the case in the past and we allow the propaga-

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tion of many transverse modes. Using numerical modeling, we look closely at the behavior of the modes in the Stokes wave, concentrating on the mode conjugate to the pump mode (i.e. phase conjugation). In the following sections we discuss the process of SBS in general, and then present the results of numerical simulations that lend new insights into the process of SBS.

2. Theory

To develop the formalism we assume a homogeneous, nonlinear, time-independent medium. The wave equation for an electric field travelling in this medium is given by

$$\nabla^2 \tilde{E} - \frac{\varepsilon}{c^2}\frac{\partial^2 \tilde{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \tilde{P}}{\partial t^2}, \tag{1}$$

where as usual ε is the dielectric constant of the medium and c is the speed of light.

Let us assume that the electric field in the medium consists of a pump wave and a Stokes wave that are generally counter propagating. Thus the field in the medium is adequately described by two counter-propagating waves with independent and slowly varying envelopes that are functions of all three spatial coordinates:

$$\tilde{E} = (E_p(\mathbf{r}, z)e^{-i(kz+\omega_p t)} + E_s(\mathbf{r}, z)e^{i(kz-\omega_s t)}) + \text{c.c.} \tag{2}$$

We now write the steady-state driven wave equation for the field described by Eq. (2). We make the slowly varying envelope approximation and separate the portions of the nonlinear polarization that are phase matched to the two parts of the electric field and treat them as independent driving terms. This allows us to write two coupled wave equations, each one describing the propagation of one of the constituent fields. These equations are

$$\nabla_{\mathrm{T}}^2 E_p - 2ik\frac{\partial E_p}{\partial z} = \frac{-4\pi\omega^2}{c^2} P_{-ikz}^{\mathrm{NL}} \tag{3a}$$

and

$$\nabla_{\mathrm{T}}^2 E_s + 2ik\frac{\partial E_s}{\partial z} = \frac{-4\pi\omega^2}{c^2} P_{+ikz}^{\mathrm{NL}}, \tag{3b}$$

where $P_{\pm ikz}^{\mathrm{NL}}$ refers to the phase matched portion of the nonlinear polarization, we assume that $\omega_p \approx \omega_s = \omega$, and ∇_{T}^2 is the transverse portion of the Laplacian operator given by $\nabla_{\mathrm{T}}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

We now consider a modal decomposition of the transverse components of the pump and Stokes fields. For the moment we will leave the basis set arbitrary, but require that the individual functions of the basis set be orthonormal and individually satisfy the homogeneous wave equation. Thus we may write the pump and the Stokes waves as a superposition of the basis sets A and B:

$$E_p(\mathbf{r}, z) = \sum_\alpha a_\alpha(z)A_\alpha(\mathbf{r}, z) \tag{4a}$$

and

$$E_s(\mathbf{r}, z) = \sum_\alpha b_\alpha(z)B_\alpha(\mathbf{r}, z). \tag{4b}$$

Substituting Eqs. 4(a) and 4(b) into Eqs. (3a) and (3b), and utilizing the fact that both A_α and B_α are solutions to the homogeneous wave equation, the coupled equations describing the propagation of the wave described by Eq. (2) become

$$2ik \sum_\alpha A_\alpha \frac{\partial a_\alpha}{\partial z} = \frac{4\pi\omega^2}{c^2} P^{\mathrm{NL}}_{-ikz} \tag{5a}$$

and

$$2ik \sum_\alpha B_\alpha \frac{\partial b_\alpha}{\partial z} = \frac{-4\pi\omega^2}{c^2} P^{\mathrm{NL}}_{+ikz} \,. \tag{5b}$$

The phase matched portion of the nonlinear polarization under the slowly varying envelope approximation is well known and, assuming on-resonance SBS, is given by[2,3]

$$P^{\mathrm{NL}}_{\mathrm{SBS}} = i6\chi^{(3)}_{\mathrm{SBS}} \left(|E_s(\mathbf{r},z)|^2 E_p(\mathbf{r},z) - |E_p(\mathbf{r},z)|^2 E_s(\mathbf{r},z) \right) \,, \tag{6}$$

where $\chi^{(3)}_{\mathrm{SBS}}$ is the contribution of the on-resonance Brillouin interaction to $\chi^{(3)}$.

By substituting Eqs. (4) into Eq. (6) we may write the phase matched portions of the nonlinear polarization in terms of the two basis sets used for the decomposition of the electric fields. Equations (5) may then be written as

$$\sum_\alpha A_\alpha \frac{\partial a_n}{\partial z} = \frac{12\pi\omega}{\sqrt{\varepsilon}c} \chi^{(3)}_{\mathrm{SBS}} \sum_{jkl} a_j A_j b_k^* B_k^* b_l B_l \tag{7a}$$

and

$$\sum_\alpha B_\alpha \frac{\partial b_n}{\partial z} = \frac{12\pi\omega}{\sqrt{\varepsilon}c} \chi^{(3)}_{\mathrm{SBS}} \sum_{jkl} b_j B_j a_k^* A_k^* a_l A_l \,. \tag{7b}$$

We now multiply each of Eq. (7) by a single member of the applicable basis set (A_n^* or B_n^*) and integrate over all of transverse space, utilizing the orthonormality of the functions to produce the equations

$$\frac{\partial a_n}{\partial z} = \frac{12\pi\omega}{\sqrt{\varepsilon}c} \chi^{(3)}_{\mathrm{SBS}} \sum_{jkl} a_j b_k^* b_l \int_{-\infty}^{\infty} d^2r A_j B_k^* B_l A_n^* \tag{8a}$$

and

$$\frac{\partial b_n}{\partial z} = \frac{12\pi\omega}{\sqrt{\varepsilon}c} \chi^{(3)}_{\mathrm{SBS}} \sum_{jkl} b_j a_k^* a_l \int_{-\infty}^{\infty} d^2r B_j A_k^* A_l B_n^* \,. \tag{8b}$$

To continue the development of the model it is now necessary to choose basis sets for the decomposition; we have chosen the Hermite-Gaussian functions as the basis set. The Hermite-Gaussian functions are chosen instead of the Laguerre-Gaussian functions for two reasons. First, the use of Hermite-Gaussian functions introduces some parity considerations into the equations that are of practical use. Second, the choice of Hermite-Gaussian modes allows the convenient modeling of aberrated beams that are not cylindrically symmetric; while not reported here, we will report the modeling of aberrated beams in the near future.

In one of the two transverse dimensions the normalized Hermite-Gaussian modes for the pump wave are given by[12]

$$A_n(x,z) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} (2^n n! w(z))^{-\frac{1}{2}} e^{i(n+\frac{1}{2})\psi(z)} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \exp\left[-i\,\frac{kx^2}{2R(z)} - \frac{x^2}{w(z)^2}\right]. \tag{9a}$$

For the Stokes wave we choose the basis set made up of the complex conjugates of the modes of the pump wave. These are given by

$$B_n(x,z) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} (2^n n! w(z))^{-\frac{1}{2}} e^{-i(n+\frac{1}{2})\psi(z)} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \exp\left[i\,\frac{kx^2}{2R(z)} - \frac{x^2}{w(z)^2}\right]. \tag{9b}$$

Here $w(z), R(z)$ and $\psi(z)$ have their usual meanings of spot size, radius of curvature of the wave front and Guoy phase angle respectively, given by

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_R}\right)^2}\,,$$

$$R(z) = z + \frac{z_R^2}{z}$$

and

$$\psi(z) = \tan^{-1}\left(\frac{z}{z_R}\right).$$

As usual, w_0 is the spot size at the beam waist and z_R is the Rayleigh range given by

$$z_R = \frac{\pi w_0^2}{\lambda}.$$

Since the transverse coordinates are orthogonal, in two dimensions the basis sets are given by

$$A_{n,m}(\mathbf{r},z) = A_n(x,z)A_m(y,z)$$

and

$$B_{n,m}(\mathbf{r},z) = B_n(x,z)B_m(y,z).$$

For notational simplicity we will develop the rest of the model in only one of the two transverse dimensions.

Having chosen a basis set, the product under the integral in Eq. (8) can be written explicitly as

$$\begin{aligned} A_j B_k^* B_l A_n^* = {} & \left(\frac{2}{\pi}\right)(2^{j+k+l+n} j!k!l!n!)^{-1/2}\left(\frac{1}{w(z)}\right)^2 \\ & \times H_j\left(\frac{\sqrt{2}x}{w(z)}\right) H_k\left(\frac{\sqrt{2}x}{w(z)}\right) H_l\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \\ & \times \exp\left(\frac{-4x^2}{w(z)^2} + i[j+k-l-n]\psi(z)\right) \end{aligned} \tag{10a}$$

and

$$B_j A_k^* A_l B_n^* = \left(\frac{2}{\pi}\right)(2^{j+k+l+n} j!k!l!n!)^{-1/2}\left(\frac{1}{w(z)}\right)^2 \times H_j\left(\frac{\sqrt{2}x}{w(z)}\right) H_k\left(\frac{\sqrt{2}x}{w(z)}\right) H_l\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \times \exp\left(\frac{-4x^2}{w(z)^2} - i[j+k-l-n]\psi(z)\right). \tag{10b}$$

Since the integral is over the transverse dimensions and the Hermite-Gaussian functions are separable, we can perform the integration in Eqs. (8) explicitly and define an overlap integral by

$$\xi_{jkln} = \left(\frac{2}{\pi}\right)(2^{j+k+l+n} j!k!l!n!)^{-1/2} \times \int_{-\infty}^{\infty} dx' H_j(\sqrt{2}x')H_k(\sqrt{2}x')H_l(\sqrt{2}x')H_n(\sqrt{2}x')\exp(-4x'^2), \tag{11}$$

where

$$x' = \frac{x}{w(z)}.$$

Equations (8) can now be written in final form as

$$\frac{\partial a_n}{\partial z} = \left(\frac{12\pi\omega}{\sqrt{\varepsilon}c}\right)\left(\frac{1}{w(z)}\right)\chi^{(3)}_{\mathrm{SBS}} \sum_{jkl} a_j b_k^* b_l \xi_{jkln} e^{i(j+k-l-n)\psi(z)} \tag{12a}$$

and

$$\frac{\partial b_n}{\partial z} = \left(\frac{12\pi\omega}{\sqrt{\varepsilon}c}\right)\left(\frac{1}{w(z)}\right)\chi^{(3)}_{\mathrm{SBS}} \sum_{jkl} b_j a_k^* a_l \xi_{jkln} e^{-i(j+k-l-n)\psi(z)}. \tag{12b}$$

Equations (12) are the steady-state coupled SBS equations in two dimensions. In three dimensions this system of equations is given by

$$\frac{\partial a_{n,m}}{\partial z} = \left(\frac{12\pi\omega}{\sqrt{\varepsilon}c}\right)\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\mathrm{SBS}} \times \sum_{jklpqr} a_{j,p} b_{k,q}^* b_{l,r} \xi_{jkln} \xi_{pqrm} e^{i(j+p+k+q-l-r-n-m)\psi(z)} \tag{13a}$$

and

$$\frac{\partial b_{n,m}}{\partial z} = \left(\frac{12\pi\omega}{\sqrt{\varepsilon}c}\right)\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\mathrm{SBS}} \times \sum_{jklpqr} b_{j,p} a_{k,q}^* a_{l,r} \xi_{jkln} \xi_{pqrm} e^{-i(j+p+k+q-l-r-n-m)\psi(z)}. \tag{13b}$$

There are three important things to note in Eqs. (13). The first thing to note is that Stokes modes that are conjugate to the pump modes will see higher gain than those that are not. That is, whenever there is an instance where $j = k = l = n$ or $p = q = r = m$, the value of the overlap integral is maximum. Stokes modes that are not conjugate to modes that exist in the pump beam have a reduced value of the overlap integral, and therefore see less gain than those modes whose conjugate does exist in the pump beam.

The second important conclusion that can be drawn from an examination of Eqs. (13) is that there are quite obviously some phase-matching requirements. These phase matching requirements can also help to ensure that modes correlated with the pump wave are amplified more than uncorrelated modes.

The phase matching requirements exhibited in Eqs. (13) are due to the propagation of the fields and not due to an overall phase matching requirement. This may be seen by substituting Eq. (6) into Eq. (3). With this substitution, the coupled amplitude equations may be written as

$$\nabla_{\mathrm{T}}^2 E_p - 2ik\,\frac{\partial E_p}{\partial z} = \frac{-i24\pi\omega^2}{c^2}\,\chi_{\mathrm{SBS}}^{(3)}|E_s|^2 E_p \tag{14a}$$

and

$$\nabla_{\mathrm{T}}^2 E_s + 2ik\,\frac{\partial E_s}{\partial z} = \frac{i24\pi\omega^2}{c^2}\,\chi_{\mathrm{SBS}}^{(3)}|E_p|^2 E_s\,. \tag{14b}$$

Equations (14) clearly show that the process of SBS is insensitive to an overall phase shift in either the Stokes or the pump beam and therefore SBS is considered a process that is automatically phase matched; however, in the formalism presented above it is clear that upon propagation there are definite phase terms that become important. The importance of these phase mismatched terms will be explored in Sec. 3.

The third important thing to note in Eqs. (13) is that every Stokes mode is coupled to every other Stokes and pump mode through the Brillouin nonlinearity. Therefore all modes affect the phase and amplitude of all other modes. In the case of SBS for optical phase conjugation, this coupling between modes results in the imposition of noise on the conjugate modes due to the presence of nonconjugate modes. As will be shown later, when the amplitude of the conjugate mode dominates, this noise can still be seen in the phase of the mode.

One of the advantages of casting the SBS process in the form of Eqs. (13) is that it is relatively easy to obtain a numerical solution provided that the process being investigated can be described by only a few members of the basis set. One example of interest to the scientific community is SBS of focused Gaussian beams. In the following section we will address the numerical solution to Eqs. (13) when the pump beam can be described by the lowest order Gaussian mode.

3. Modeling

In this section we present results of the numerical solutions to Eqs. (13) under conditions often encountered in the laboratory. In particular, the beams modeled are Gaussian in intensity and focused into the Brillouin medium.

In modeling the SBS process we numerically solve Eqs. (13) with one slight modification. We assume that there is no coupling between modes in orthogonal transverse dimensions higher than order zero. That is, we allow the exchange of energy between all modes of one dimension and the fundamental mode of the orthogonal dimension. However, we do not consider coupling between higher order modes in orthogonal dimensions. This is equivalent to assuming that the SBS process treats each transverse dimension independently for modes of order greater than zero. This is not strictly true, but since most of the power in both transverse dimensions resides in the fundamental Gaussian mode we take this as a good approximation. Additionally we find the results of the simulation to accurately describe the Stokes beam produced by a Gaussian pump beam in the laboratory.[13]

Understanding that the single index on the coefficients implies the lowest order mode in the orthogonal dimension, we will define the SBS coupling constant by

$$K = \frac{12\pi\omega}{\sqrt{\varepsilon}c}\,\xi_{0000}\,. \tag{15}$$

Equations (13) may then be written in the form used for our numerical analysis as

$$\frac{\partial a_n}{\partial z} = K\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\mathrm{SBS}}\sum_{jkl} a_j b_k^* b_l \xi_{jkln} e^{i(j+k-1-n)\psi(z)} \tag{16a}$$

and

$$\frac{\partial b_n}{\partial z} = K\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\mathrm{SBS}}\sum_{jkl} b_j a_k^* a_l \xi_{jkln} e^{-i(j+k-1-n)\psi(z)}\,, \tag{16b}$$

where the possibilities of modeling an aberrated beam is left open by allowing for astigmatic modes; this is shown explicitly by writing the spot sizes of the orthogonal transverse dimensions separately.

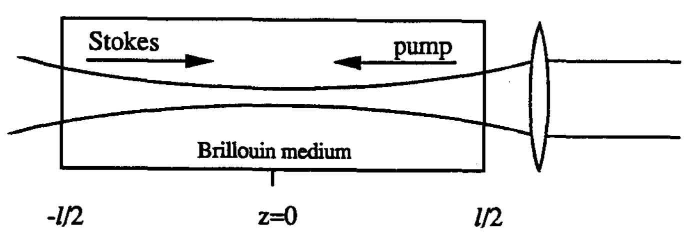

Fig. 1. The physical geometry and the coordinate system assumed in the model.

The coordinate system and physical picture used for the modeling are shown in Fig. 1. The choice of the origin at the focus of the pump beam rather than one end of the Brillouin medium is made to facilitate the modal decomposition. The modes used as a basis set for the decomposition are completely specified by Eqs. (9), provided we know the spot size at the beam waist w_0. Since we are interested in investigating focused Gaussian beams, we have defined the basis set by the spot size at the focus of the pump beam. The spot size at the focus of a lens of focal length f may be related to the spot size at the entrance to the focusing lens by[12]

$$w_0 = \frac{f\lambda}{\pi w(f)}\,. \tag{17}$$

In order to study the effects of stochastic initiation we have added noise to the simulation. Physically the noise is due to spontaneous Brillouin scattering and is responsible for the initiation of the SBS process. It has been shown that the intensity of noise due to spontaneous Brillouin scattering is on the order of 10^{-12} the intensity of the pump.[2,3] In order to introduce noise into the simulation, at each point in the medium a Stokes wave was introduced that had random phase with amplitude equal to 10^{-6} of the pump at that point in the medium. Thus some of the effects of distributed noise could be studied in the context of the steady-state SBS process.

The ability to numerically model the SBS process using the formalism developed in Sec. 2 relies on choosing a geometry that can be described by only a few modes of the basis set. We have found that under most situations of interest in our research, ten Hermite-Gaussian modes is an adequate number to describe the stimulated Brillouin scattering resulting from a pump that is initially described by a Gaussian beam of lowest order. Fortunately, due to parity considerations the growth of the odd numbered modes is very small compared to the even numbered modes, and therefore in many cases it is only necessary to explicitly integrate the first five even numbered modes to accurately predict the behavior of the fields in a Brillouin medium in a focused geometry for a variety of input powers.[13]

The spatial distribution of the power in the first two even numbered modes of the Stokes wave for an incident focused Gaussian beam is shown in Fig. 2. Also shown in Fig. 2 is the evolution of the power in the mode of the incident beam that was input into the medium (the A_0 mode). The phase-conjugating ability of SBS is clearly seen by the dominance of the B_0 mode in the Stokes beam. In this case the simulation was conducted with the first ten Hermite-Gaussian modes, but the power in the remaining modes of the basis set is too small to be of any significance on the scale shown in Fig. 2. In Fig. 2 the abscissa is written in terms of the Rayleigh range of the incident beam and the ordinate is normalized to the input power of the pump. The situation modeled is that shown in Fig. 1 with the Stokes beam containing 88% of the input power.

The phase-conjugate fidelity of the SBS process is found by dividing the power in the mode conjugate to the pump by the total power in the Stokes beam. In the

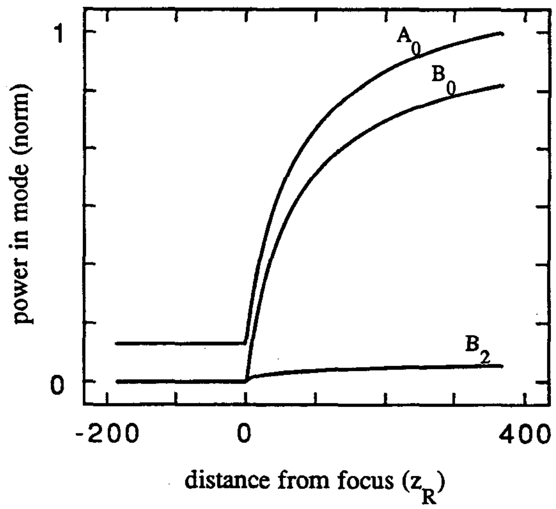

Fig. 2. Power in the A_0 mode of the pump beam, and the B_0 and B_2 modes of the Stokes beam as a function of distance within the Brillouin medium during SBS. The situation modeled is for ten times above threshold.

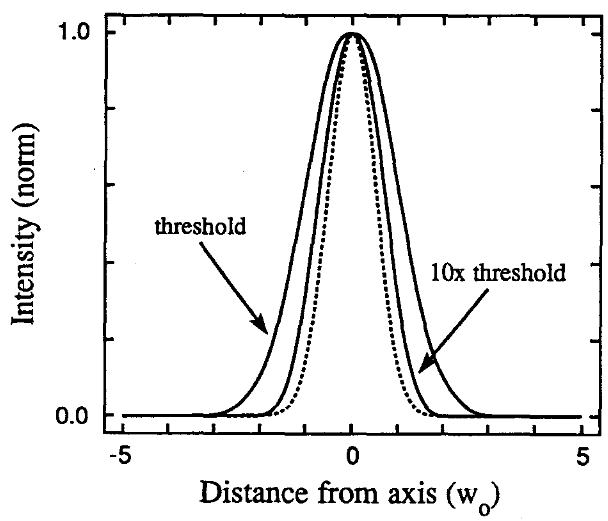

Fig. 3. The intensity of the Stokes beam as it leaves the medium, calculated at ten times the SBS threshold and at SBS threshold. The dashed line is the profile of the pump beam while the solid lines are those of the Stokes beams. The units on the abscissa are multiples of the pump beam diameter. In both cases the Stokes beam is an almost perfect Gaussian with diameter 1.9 (at threshold) and 1.3 (ten times threshold) times that of the pump.

situation shown in Fig. 2 the phase-conjugate fidelity is 0.95. Figure 3 shows the near-field profile of the Stokes beam as it leaves the medium for both the case of that shown in Fig. 2 and for the case near SBS threshold (SBS reflectivity of 0.10, phase-conjugate fidelity of 0.80). The shape of the Stokes beam is almost perfectly Gaussian in both instances, but the beam is larger than the pump beam at the entrance to the medium.

In many experimental investigations the Stokes beam passes back through the lens focusing the pump beam into the medium and is then reflected out of the pump beam path for analysis. The phase-conjugate fidelity is measured in the far-field by focusing the Stokes beam onto a pinhole whose size is chosen to be the diffraction-limited size of the pump beam. Thus a Stokes beam exactly matching the pump beam would pass as much power through the pinhole as the pump beam and a fidelity of unity would be measured. Unfortunately, in the case of a beam that is wider than the pump but still Gaussian in shape, the fidelity measured will be in excess of unity. This has been seen in experiments[14] and would be the case of a real beam similar to that modeled here. Thus, while the fidelity of the Stokes beam near threshold is calculated as 0.80, using the method described above the measured fidelity would be approximately 1.1.

In the laboratory it appears that most Stokes beams derived from Gaussian pump beams are Gaussian in shape, but not all larger than the pump beam upon exiting the medium as our simulation predict. There are reports of Stokes beams that are narrower than the pump,[15] and it is theorized that the narrower beam width is due to axial gain pulling[16] near the entrance to the medium. In our simulations we see the results of axial gain pulling at and near the focal point of the pump beam within the Brillouin medium, but to produce a beam narrower than the pump beam axial gain pulling must occur well away from the focal point. In our simulations, we do not detect the effects of axial gain pulling far from the focus that are required to produce a beam narrower than the pump at the entrance to the medium, even when the pump beam has a power 100 times the SBS threshold power. It is possible that self-focusing away from the focal point accounts for the reduction in beam diameter observed in the laboratory and we are currently investigating this idea by adding other nonlinear effects to the simulation.

In all of our simulations the shape of the Stokes beam exiting the medium was always an almost perfect Gaussian when the pump beam was a Gaussian. These results are in contrast to other numerical modeling efforts to predict the behavior of SBS well above threshold, where the predicted Stokes profiles are distinctly not Gaussian.[17]

There are two parameters that characterize plots of the type shown in Fig. 2. The first is the length of the medium relative to the Rayleigh range of the incident beam. This parameter has a large effect both on the behavior of the power transfer between the incident and Stokes beams and on the ability of the process to produce a phase-conjugate of the incident wave. By using the Rayleigh range of the pump beam as the length parameter in the simulation, the output is independent of the Brillouin medium or the details of the pump beam focusing. Therefore as long as the ratio of the Rayleigh range to the length of the medium remains the same, the results will be the same regardless of the actual medium that is being considered.

The second important parameter is the power in the incident beam. That is, it is important to know by how much the incident beam power exceeds the threshold for SBS. It is common to quote the power in the beam in units of the threshold power for SBS. What is implicit in stating the power in this manner is an understanding of what the SBS threshold is; unfortunately there is no consensus in the literature as to what exactly defines SBS threshold.

For theoretical investigations one normally assumes that SBS threshold occurs at the point where the product of the Brillouin gain and the on-axis pump intensity integrated over the length of the medium reaches a certain value (usually ~25–30). For engineering expediency the SBS threshold is often defined as the point at which the Stokes power leaving the medium reaches 10 % of the value of the incident beam power. While there are several disadvantages to this latter definition, the "10 % criteria" has the advantage of being directly measurable in the laboratory. As long as one ignores the semantic problem of being able to have SBS occur below SBS threshold, the 10 % criteria is quite useful and we will therefore adhere to that convention. The interaction shown in Fig. 2 is for an incident beam that is ten times over threshold.

From Fig. 2 it is obvious that most of the power is extracted from the incident beam in the region in front of the focus. However, SBS is actually the result of the selective amplification of noise within the medium and must be viewed from that perspective.

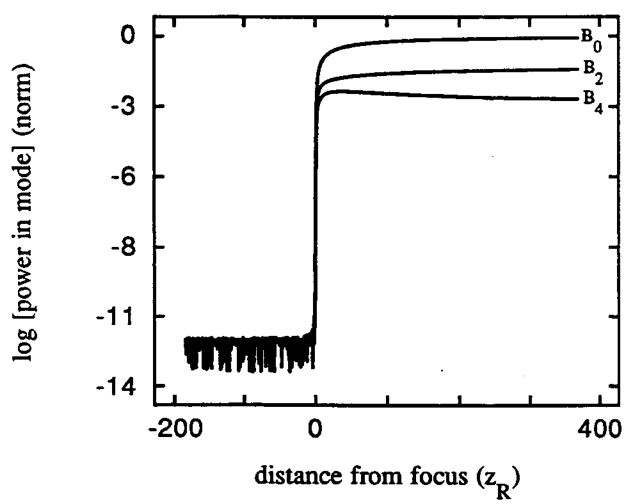

Fig. 4. Plot of the log of the power in the first three even modes of the Stokes wave showing the amplification of noise and discrimination against nonconjugate modes. The incident beam was made up entirely of the A_0 mode.

In order to demonstrate the noise amplification aspect of SBS, in Fig. 4 we have plotted on a semi-log scale the first three even modes of the Stokes beam for the same situation shown in Fig. 2. Figure 4 clearly shows that almost all of the amplification of the initiating noise occurs at or near the focus and that very little actually occurs away from the focal region; this is of course expected from an intuitive understanding of SBS. Furthermore, Fig. 4 shows most of the discrimination against nonconjugate modes occurs at or very near the focus.

The inclusion of distributed noise in our model is a crucial aspect of our investigation. Some recent research has focused on the temporal aspects of the distributed nature of the noise in SBS,[18,19] but little attention has been given to the spatial aspects.

In our model the amplitude of the noise in the Stokes wave at every point in the medium is fixed at 10^{-6} of the pump amplitude at that point, and given a random phase. By using a distributed noise source we may investigate exactly where SBS starts in the focused geometry and what regions are important in the development of the phase of the Stokes beam.

Note that we have chosen to keep the noise amplitude fixed rather than use the more realistic model of a random amplitude with a mean of 10^{-6} of the pump. We do this so that conclusions about the SBS initiation process are not clouded by normal statistical fluctuations. Under ideal circumstances one would wish to use a stochastic phase and Gaussian amplitude distribution. However, given the length of time required for one simulation, it would be impractical to perform enough simulations to ensure that normal statistical fluctuations were not biasing the conclusions. Therefore we use a constant amplitude for the noise and follow the effects of the noise by examining the phase of the wave.

The effects of the noise on the amplitude of the Stokes beam are clearly seen in the portion of Fig. 4 where $z < 0$, but it is the phase of the Stokes beam that yields the most information about the initiation process. Figure 5 shows the phase of the lowest order mode of the Stokes beam far below SBS threshold and at ten times threshold (only every fifth point has been plotted for purposes of clarity). The arrow in Fig. 5a indicates the phase of the noise at the end of the Brillouin medium.

Figure 5b demonstrates that even very high above SBS threshold, the phase of the rear-most point determines the phase of the Stokes beam; we find this to be true regardless of the noise seed or the pump power (up to 100 times threshold). From a naive point of view one would expect this to be true, since the integral of the product of the gain and the intensity over the length of the medium is the greatest if one begins at the farthest possible distance from the front of the medium. It is surprising however, that even ten times above threshold the phase of the Stokes beam is determined by the noise at the rear-most point in the medium, a point where the pump intensity (and hence the intensity of the spontaneous Stokes scattering) is orders of magnitude lower than near the focus. Judging from the evolution of the amplitude shown in Fig. 4, one may have thought that the initiating noise occurred much closer to the focus when the pump is far above threshold and indeed there have been predictions of this effect,[20] but we have found no case in our simulations where this is true. Of course had intensity fluctuations been included in our model, the

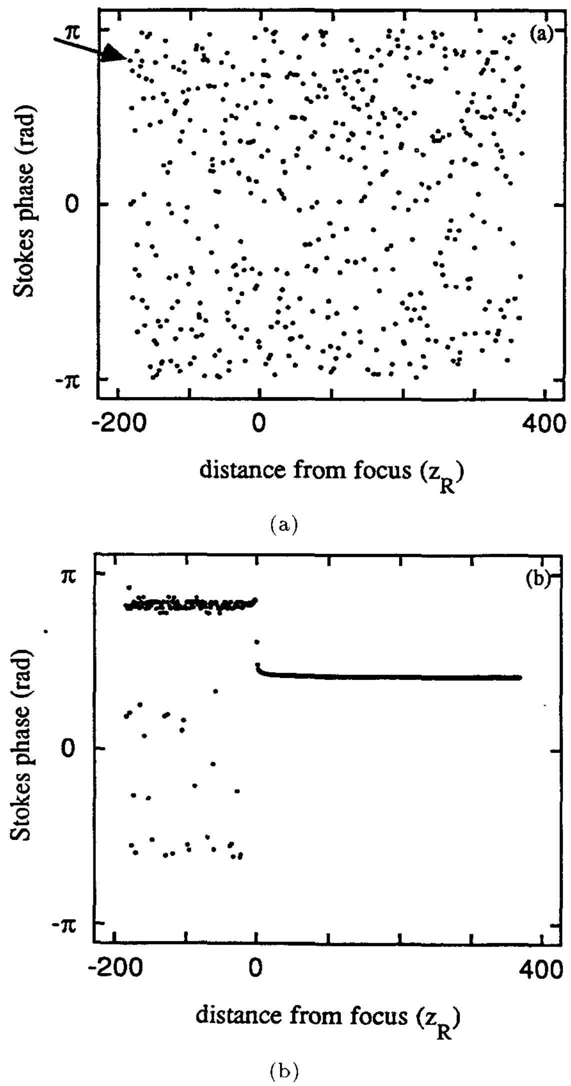

Fig. 5. The phase of the lowest order mode of the Stokes radiation far below SBS threshold (a) and ten times above SBS threshold (b). The arrow in (a) indicates the phase of the noise at the point at the back end of the Brillouin medium.

point of initiation could have been slightly different, since a peak in the intensity of the noise could dominate over the length factor. However, assuming that the amplitude of the spontaneous scattering is proportional only to the amplitude of the pump wave at every point in the medium, the point of initiation does not appear to change regardless of the power in the pump beam.

Thus, the effects of noise in a given model, other than at the point of initiation, are not significant in determining the phase of that mode in Stokes beam. However, the phase of a given Stokes mode can be influenced by other modes in the Stokes beam. This effect is seen in Fig. 5b, where the phase of the coefficient of the Stokes wave undergoes a phase shift as it propagates through the focus. The magnitude of this phase shift depends in detail on the phase and magnitude of the other Stokes modes and varies with the intensity of the pump. If no modes other than the mode conjugate to the pump are allowed to propagate in the simulation, the coefficient of the Stokes wave undergoes no phase shift at all.

The origin of this phase shift is the coupling between modes mentioned in Sec. 2. An examination of Eqs. (13) reveals that even when one mode is dominant, the phases and amplitudes of all modes (including the dominant mode) are dependent upon the phases and amplitudes of all of the other modes. The more dominant a single mode is, the less effect the coupling between modes will have on the phase and amplitude of the dominant mode. In the case of the pump wave, where the mode structure is imposed externally, the phase shifts of the coefficients of the pump mode are very small upon propagation through the medium.

In the case considered in Fig. 5b, the phase of the coefficient of the Stokes wave moves in the same direction as the Guoy phase shift. Thus the Stokes wave has a total phase shift on the order of $3\pi/2$ as it propagates from the back of the medium to the front.

To demonstrate the effect that noise modes in the Stokes beam have on the mode conjugate to the pump wave we have solved Eq. (16) for a variety of different initial phases of the nonconjugate Stokes modes, keeping the initial phase of the conjugate mode fixed. We do this by forcing both the phase and the amplitude of the initiating radiation to be fixed for each mode throughout the medium. The noise in the B_0 Stokes mode was fixed with zero phase and amplitude 10^{-6} of the pump. The noise in the other Stokes modes was fixed at 10^{-6} of the value of the pump in amplitude, but the phase was varied from zero to 2π. The results of these simulations are shown in Fig. 6. In Fig. 6 the phase of the coefficient of the mode conjugate to the pump mode (B_0 mode) is plotted as a function of the initial phase of the other Stokes modes. It is clear from Fig. 6 that even though the B_0 mode dominates the Stokes wave ($\sim$80% of the power is in the B_0 mode), its phase is determined in large part by the amplitude and phase of all the other Stokes modes. Naturally, had the pump wave contained modes other than A_0 mode, they would have affected the phase and amplitude of the Stokes wave as well.

It is surprising that the nonconjugate, non-phase matched modes could have such a large effect on the Stokes wave. However, a similar behavior is observed for all modes in the region of high Brillouin gain. That is, a phase shift is observed at the focal point, and the magnitude of that shift is determined by the magnitudes and phases of all of the other modes allowed to propagate.

A careful study of these phase shifts shows that they are the result of all of the modes attempting to shift their phase to maximize the gain. That is, the phase of each mode is shifted to maximize the constructive effect of all of the non-phase matched terms. Since all of the non-phase matched terms are also combined with a reduced overlap integral, one could wonder whether the effect of the non-phase

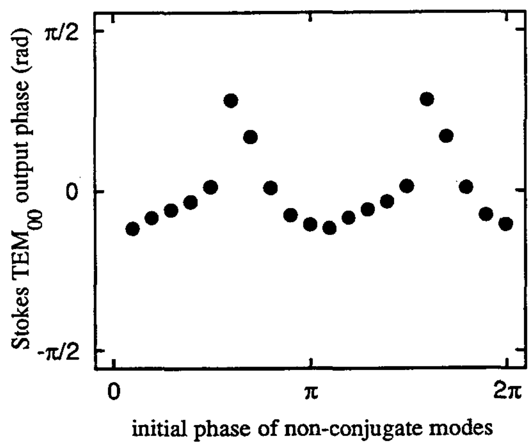

Fig. 6. Demonstration of the effects of noise modes on the phase of the Stokes mode that is conjugate to the pump. The phase of the initiating radiation was held constant for the B_0 mode and varied for the phase of the nonconjugate modes.

matched terms is of much importance. Our research has shown that the non-phase matched terms are extremely important.

To demonstrate the importance of the non-phase matched terms in determining the SBS gain of any given mode we will examine the B_0, B_1 and B_2 modes of the Stokes beam in the simple case of a lowest order Gaussian beam focused into a Brillouin medium. First let us assume that only the A_0 mode of the pump beam is allowed to propagate, and then examine the first three terms in each of the differential equations describing the first three Stokes modes.

The three differential equations are:

$$\frac{\partial b_0}{\partial z} = K\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\mathrm{SBS}}|a_0|^2\{b_0\xi_{0000} + b_1\xi_{1000}e^{-i\psi(z)} + b_2\xi_{2000}e^{-2i\psi(z)}\} \tag{18a}$$

$$\frac{\partial b_1}{\partial z} = K\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\mathrm{SBS}}|a_0|^2\{b_0\xi_{0001}e^{i\psi(z)} + b_1\xi_{1001} + b_2\xi_{2001}e^{-i\psi(z)}\} \tag{18b}$$

$$\frac{\partial b_2}{\partial z} = K\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\mathrm{SBS}}|a_0|^2\{b_0\xi_{0002}e^{2i\psi(z)} + b_1\xi_{1002}e^{i\psi(z)} + b_2\xi_{2002}\}\,. \tag{18c}$$

To understand these equations it is important to know the values of the overlap integral ξ_{jkln}. The values of the overlap integrals in Eqs. (18) are:

$$\xi_{0000} = 0.56$$
$$\xi_{2000} = -0.20$$
$$\xi_{1001} = 0.28$$
$$\xi_{2002} = 0.21\,,$$

with all of the others being zero due to parity considerations.

Replacing the overlap integrals with their actual values produces the following set of coupled equations:

$$\frac{\partial b_0}{\partial z} = K\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\text{SBS}}|a_0|^2\{(0.56)b_0 - (0.20)b_2 e^{-2i\psi(z)}\}\,, \tag{19a}$$

$$\frac{\partial b_1}{\partial z} = K\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\text{SBS}}|a_0|^2\{(0.28)b_1\}\,, \tag{19b}$$

$$\frac{\partial b_2}{\partial z} = K\left(\frac{1}{w_x(z)}\right)\left(\frac{1}{w_y(z)}\right)\chi^{(3)}_{\text{SBS}}|a_0|^2\{(-0.20)b_0 e^{2i\psi(z)} + (0.21)b_2\}\,. \tag{19c}$$

Examination of Eqs. (19) would lead one to believe that the lowest order mode would dominate a rigorous integration of the SBS equations under these conditions. The value of the overlap integral for the phase matched term of the B_0 mode is twice that of the other two modes and would see the highest gain. The B_1 coefficient has the next highest gain and the B_2 coefficient sees the least gain of the three. In this case of a focused Gaussian beam, the phase angle is changing rapidly in the region of highest gain and therefore one would think that the importance of the non-phase matched terms is even less than indicated by the value of the overlap integral.

Numeric integration of Eq. (19) shows that indeed the B_0 mode dominates the Stokes beam as expected. The value of the B_0 mode is an order of magnitude larger than that of the next largest coefficient. However, the next largest coefficient is that of the B_2 mode; the value of the coefficient B_1 is fully three orders of magnitude below that of the coefficient B_2 when the equations are integrated well above SBS threshold. This demonstrates that the overlap of the Stokes mode and the pump mode is not the dominant factor in determining which mode sees the highest gain.

This surprising result is due to the fact that as the modes propagate, they change phase to enhance their gain through the non-phase matched terms. This may be described as *phase pulling* and is the origin of the phase shift exhibited in Fig. 5. The B_1 mode sees such little gain because the coefficient for the non-phase matched term connecting the B_1 mode to the B_0 Stokes mode is zero. In the case of only three propagating modes, there is no phase of the B_1 mode that can in any way enhance the gain over the basic phase matched term. The situation of reduced gain due to the lack of an overlap integral connecting them with the B_0 Stokes mode is the same for all of the odd numbered modes.

If instead of a pure A_0 mode pump beam there were odd numbered pump modes present, naturally there would be non-phase matched terms in the equation for the odd numbered Stokes coefficients. However, there would still be no coupling between even and odd numbered Stokes modes except through the pump beam.

The phase shifts on the Stokes modes as they propagate through a region of high gain are then due to the process of phase pulling. This process will produce an intensity dependent phase shift on each mode of the Stokes beam as the phases adjust themselves for maximum gain. In the process of SBS, this will appear as phase noise on the Stokes beam.

The process of phase pulling may be intuitively understood by noting that in the example just considered, all of the even modes have either a local maximum or a local minimum at the same transverse coordinate as the maximum of the mode conjugate to the pump (beam center). Simply by shifting the phase of the B_0 mode and/or all of the other even numbered modes, all of these inflection points may be made to add constructively; this is the process of phase pulling. If instead of the mode amplitudes one were examining the SBS process in terms of the intensity structure, the process of phase pulling would be described as gain guiding. However, the presence of the overall shift of phase of the dominant fundamental mode is not predicted by a simple gain guiding argument.

While the analysis of phase pulling provides important insight into the physics of SBS, the importance of this induced phase shift in stimulated Brillouin scattering is debatable since the absolute phase of the Stokes wave is often of little interest. However, phase pulling does add phase noise to the Stokes beam as the intensity of the pump changes since as the intensity of the pump changes, the phase of the Stokes beam changes. Phase pulling effects related to the coupling between Stokes modes may also be very important in the case of a Brillouin amplifier, where the absolute phase of the Stokes wave may be of great importance. Eventually a complete understanding of the process of phase pulling must wait until it can be addressed within the context of a model that includes the time dependent nature of the Brillouin process.

4. The Brillouin Gain Parameter

In this section we use the model described above to investigate the response of the Brillouin gain parameter to variations in the pump power.

We may write the undepleted pump solution for the Stokes wave in terms of the intensity in the familiar manner

$$I_{\rm S}(z) = I_0 e^{\eta(z)} \tag{20}$$

and define the Brillouin gain parameter as

$$\eta(z) = \int_{-l/2}^{z} g I_p(z') dz' , \tag{21}$$

where again g is the Brillouin gain, $I_p(z)$ is the intensity of the pump wave measured at the symmetry axis, and I_0 is the intensity of the spontaneous scattering in the medium. Since it is often the total value of the Brillouin gain parameter that is of interest, for notational convenience we will define

$$\eta(l/2) = G . \tag{22}$$

Note that this definition of the Brillouin gain parameter is different than that normally defined in the literature. Normally the quantity $I_p(z)$ is defined as the undepleted pump intensity; here we have used the actual (depleted) value of $I_p(z)$ in the integral.

As mentioned above, the intensity of the spontaneous scattering in the medium is on the order of 10^{-12} of the intensity of the pump wave. The stimulated Brillouin scattering intensity becomes comparable to the pump intensity when $G \sim 27$. Thus a value of $G = 25$–30 is often quoted as a theoretical threshold for SBS. The question may be asked what happens when the pump power is increased after G becomes on the order of 25? That is, what effect does pump depletion have on the Brillouin gain parameter?

The model we have developed is specifically designed to investigate the distribution of the intensity within the Brillouin medium during SBS, so it is natural to use the simulation to investigate the value of the Brillouin gain parameter throughout the medium at different intensities of the pump.

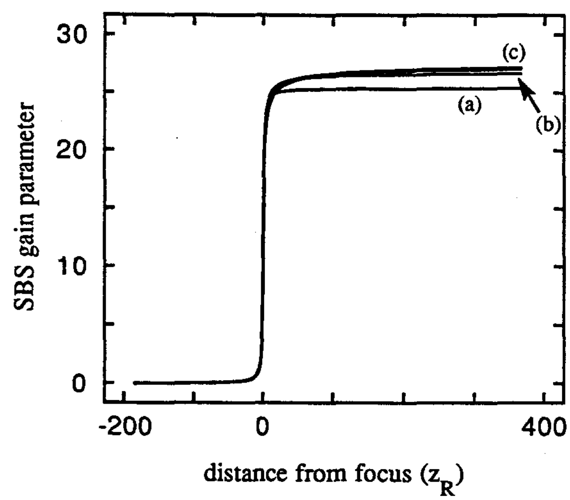

Fig. 7. Evolution of the Brillouin gain parameter $\eta(z)$ as a function of distance within the medium. The values of the pump intensity are: (a) threshold ($G = 25.37$), (b) five times above threshold ($G = 26.63$), (c) ten and 15 times above threshold ($G = 27.01$ and $G = 27.15$).

Figure 7 shows the evolution of the on-axis value of $\eta(z)$ in the Brillouin medium for four values of pump intensity ranging from SBS threshold to 15 times above threshold. The value of G for each case increased as the pump intensity was increased, but clearly not linearly. In fact, the difference between the final value of the Brillouin gain parameter for ten and 15 times above threshold (~0.5 %) is not noticeable on the scale used in Fig. 7.

Since it is obvious from Fig. 7 that G does not increase linearly with increased pump intensity, we may conclude (since the Brillouin gain, g,is a constant) that the longitudinal intensity profile of the pump wave has been modified by the SBS process. That is, above SBS threshold a significant amount of power gets extracted from the pump wave before the focal point. As the pump intensity is increased, the amount of power extracted from the pump wave by the Stokes wave at points in front of the focus becomes proportionally larger, so that an increase in incident pump intensity does not result in a proportional increase in G. With extensive modeling we have shown that the power of the waves in a focused Brillouin interaction has a characteristic functional form as a function of longitudinal coordinate z.

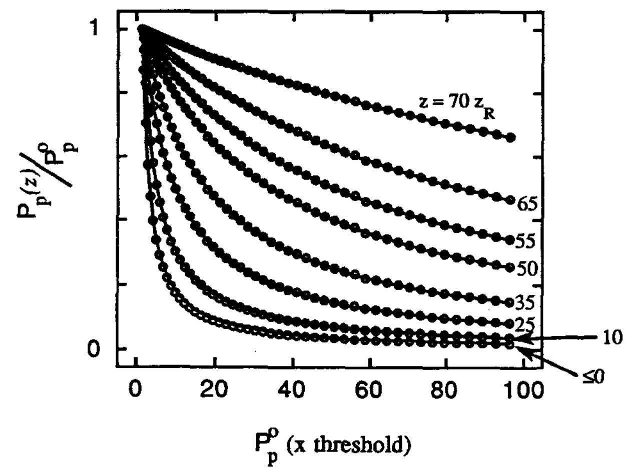

Fig. 8. Power in the pump beam at different points in the Brillouin medium plotted as a function of the input pump power. The points are derived from simulation and the lines are fit to Eq. (23).

To demonstrate the functional form of the power in the pump wave as a function of the longitudinal coordinate, in Fig. 8 we have plotted the power in the pump wave ($\mathcal{P}_p(z)$) at several different points in the medium versus the input pump power. The points in Fig. 8 are derived from simulations using the model described above and the lines are a fit to the equation.

$$\frac{\mathcal{P}_p(z)}{\mathcal{P}_p^0} = \frac{1}{a(z) + b(z)\mathcal{P}_p^0}\,, \tag{23}$$

where $\mathcal{P}_p^0 = \mathcal{P}_p(l/2)$ and $a(z)$ and $b(z)$ are constants determined by the position in the Brillouin medium. Equation (23) shows that from the perspective of the pump beam, the SBS process looks like a reverse saturable-absorber.

Equation (23) is only valid above SBS threshold and therefore the constant $a(z)$ is of little importance. The z-dependent parameter $b(z)$ in Eq. (23) is of importance however. The parameter $b(z)$ indicates the strength of the saturation of the SBS process at that point in the medium, and we will therefore refer to it as the *Brillouin saturation parameter.* The Brillouin saturation parameter is plotted in Fig. 9 as a function of position in the medium. The actual value of $b(z)$ will very with varying focal geometries, but the functional form is constant. The functional form of $b(z)$

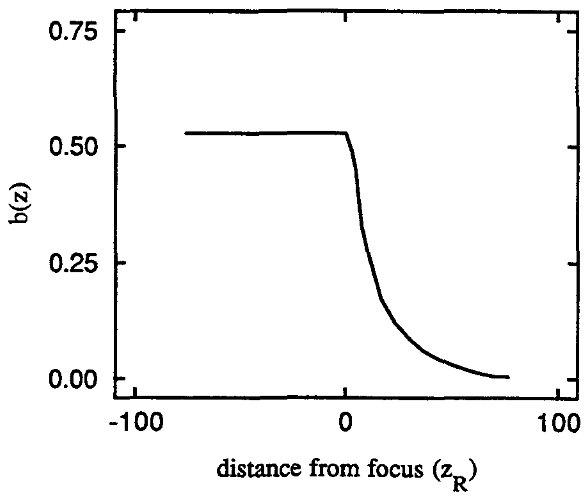

Fig. 9. The saturation parameter plotted as a function of position in the Brillouin medium. From the front of the medium to the focus $b(z)$ is almost exponential; past the focus $b(z)$ is a constant.

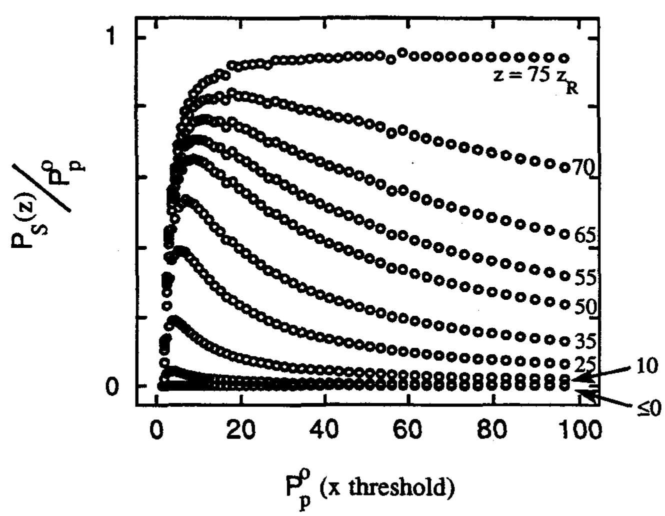

Fig. 10. Power in the Stokes beam at different points in the Brillouin medium plotted as a function of the input pump power.

is not easily fit, but we find that the form is almost exponential from the entrance to the medium to the focal point, where it becomes a constant.

Unfortunately, the functional form of the Stokes wave as a function of position and pump power is not so easy to analyze. As yet we have found no simple, universal dependence of the power in the Stokes wave on the parameter $\mathcal{P}_p^0$. A plot of the normalized power in the Stokes wave as a function of $\mathcal{P}_p^0$ and position in the medium is shown in Fig. 10.

5. Conclusions

We have reported here a detailed investigation of the process of SBS with focused Gaussian beams. This investigation was possible through the development and use of a three-dimensional simulation of the SBS process. We have investigated some of the factors that affect the creation of a Stokes beam and have reported the discovery of some interesting aspects of SBS. Much of the new insight comes from considering transverse effects that are often ignored during analysis of the SBS process.

In addition to demonstrating the limited phase conjugation of single mode pump beams, we have shown that spontaneous noise at points other than where the initiation of SBS occurs does not significantly effect the SBS process; however, the phase of a given Stokes mode is affected by the presence of other modes. The presence of other modes in the Stokes beam results in a shift in the phase of all Stokes modes, a process we have termed *phase pulling*. Phase pulling demonstrates that the overlap between a Stokes mode and the pump mode may not be the most significant factor in determining the gain of a given mode. Indeed, the nonlinear coupling between modes within the Stokes and pump beams, via the Brillouin interaction, is the dominant factor in the cases we have studied.

Our simulations have also shown that for a focused Gaussian beam the Brillouin gain parameter does not increase linearly with increased pump power once SBS threshold is reached. Rather, above threshold the SBS process mathematically resembles a reverse saturable absorber for the pump wave at any given point in the medium between the entrance to the medium and the focal point of the pump. To characterize this effect for different media and focal geometries we have defined the *Brillouin saturation parameter*. While difficult to characterize exactly, the Brillouin saturation parameter is almost exponential from the entrance to the medium to the focus of the pump beam.

References

1. I. L. Fabelinskii, *Molecular Scattering of Light* (Plenum Press, New York, 1968).
2. B. Ya. Zel'dovich, N. F. Pilipetsky and V. V. Shkunov, *Principles of Phase Conjugation* (Springer-Verlag, Berlin, 1985).
3. R. W. Boyd, *Nonlinear Optics* (Academic Press, San Diego, 1992).
4. V. G. Sidorovich, *Sov. Phys. Tech. Phys.* **21**, 1270 (1976).
5. I. M. Bel'dyugin, M. G. Galushkin, E. M. Zemskov and V. I. Mandrosov, *Sov. J. Quantum Electron.* **6**, 1349 (1976).
6. N. B. Baranova and B. Ya. Zel'dovich, *Sov. J. Quantum Electron.* **10**, 172 (1980).
7. B. Ya. Zel'dovich and T. V. Yakovleva, *Sov. J. Quantum Electron.* **11**, 186 (1981).
8. I. Yu. Anikeev, I. G. Zubarev and S. I. Mikhailov, *Sov. Phys. JETP* **57**, 978 (1983).
9. P. Suni and J. Falk, *J. Opt. Soc. Am.* **B3**, 1681 (1986).
10. E. J. Miller, M. D. Skeldon and R. W. Boyd, *Appl. Opt.* **28**, 92 (1989).
11. A. Kummrow, *Opt. Comm.* **96**, 185 (1992).
12. See for example, A. E. Siegamn, *Lasers* (University Science, Mill Valley, 1976).
13. T. R. Moore, G. L. Fischer and R. W. Boyd, *Technical Digest of the 1993 Quantum Electronics Laser Science Conference* (Optical Society of America, Washington, D.C., 1993).

14. J. J. Ottusch and D. A. Rockwell, *Opt. Lett.* **16**, 369 (1991).
15. C. L. Vercimak, C. Marshall, T. R. Moore and D. Milam, *Conference on Lasers and Electro-Optics, 1989 Technical Digest Series, Vol. 11* (Optical Society of America, Washington, D.C., 1989) paper MI4.
16. N. B. Baranova and B. Ya. Zeldovich, *Sov. J. Quantum Electron.* **10**, 555 (1980).
17. H. Hsu and S. Bor, *J. Quantum Electron.* **25**, 430 (1989).
18. R. W. Boyd and K. Rzazewski, *Phys. Rev.* **A42**, 5514 (1990).
19. A. L. Gaeta and R. W. Boyd, *Phys. Rev.* **A44**, 3205 (1991).
20. N. M. Nguyen-Vo and S. J. Pfeifer, *J. Quantum Electron.* **29**, 508 (1993).