Propagation of a Single Photon in a 1-D Waveguide Coupled to a V-Type Atom

by

Saad F. Jaddua

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Department of Physics
Faculty of Science
University of Ottawa
Canada

Abstract

The area of waveguide quantum electrodynamics plays a significant role in many quantum technology applications. Photonic waveguides can be used to coherently transport photon pulses and can also efficiently increase the light-matter interaction strength.

The work done by Shen and Fan [1, 2] on the propagation of a photon pulse in a 1-D waveguide coupled to a single two-level emitter shows that a photon pulse can undergo complete reflection or transmission through the waveguide by controlling the coupling strength between the atom and the guided photon.

We propose a V-type atomic system embedded in a 1-D waveguide and study the properties of an incoming photon pulse in the waveguide. We show that the driving field, which couples one of the atomic transitions, can be used as a controlling parameter to allow the photon pulse to be reflected or transmitted through the waveguide. As the reflection and transmission of the incident photon pulse can be externally controlled in our system, in comparison to the system suggested by Shen and Fan [1, 2], the proposed system in this thesis is more practiced than the system [1, 2].

Acknowledgements

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Chapter 1

Introduction

The ability to control and manipulate the interaction between atoms and photons depends highly on the environment conditions. Photonic waveguides [3–5] are very good candidates to significantly increase the coupling between atoms and photons, which opens the door for important applications in many research areas such as quantum computing and quantum communications. Different schemes have been introduced to couple quantum emitters to photonic waveguides for various purposes such as efficient transport and control of light pulses through waveguides and photon routing [6–12].

The propagation of a single photon pulse in a 1-D waveguide coupled to a two-level atom has been proposed [2, 13–16]. The incoming photon pulse can be completely reflected when the frequency of the pulse is in resonance with the transition frequency of the atom. The coupling strength between the atom and the guided photon also plays a significant role in reflecting or transmitting the incident photon.

Exploring the propagation of a photon pulse in a 1-D waveguide coupled to different types of three-level atomic systems have been considered [17–21]. Most of the existing schemes that use three-level atoms only consider the static solution of the problem [2], which for example does not allow one to see the time-dependence of the reflectivity and transmittivity of a photon pulse.

In this thesis, we consider the dynamical theory of the propagation of a photon pulse in a 1-D waveguide coupled to a three-level atom, where one of the atomic transitions in the atom is coupled by an external coherent field. We show that the frequency of this field can be used to control the reflection and transmission of the incident photon pulse.

The thesis is organized as follows. The second chapter reviews the necessary mathematical foundations of the work presented in this thesis. The third chapter is a review of the important results of the atom-filed interaction problem in quantum optics. In the fourth chapter, we study the transport of a photon pulse in two parallel waveguides coupled to a single two-level atom. In the fifth chapter, we explore the original contribution of this thesis, the propagation of a single photon pulse in a 1-D waveguide coupled to a three-level atomic system. Finally, we give a summary of the work presented in this thesis.

Chapter 2

Theoretical Framework

This chapter briefly discusses the foundations of the electromagnetic theory both classically and quantum mechanically.

2.1 Classical Treatment of the Electric Field

In this section, we present the basic mathematical steps to get the wave equation using the electromagnetic theory, which describes the propagation of the electromagnetic field. In the absence of any source, Maxwell equations which describe the electric and magnetic field are given by [22]

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \tag{2.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{2.2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{d\boldsymbol{B}}{dt}, \tag{2.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{d\boldsymbol{E}}{dt}, \tag{2.4}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic field vectors, respectively. The constants ϵ_0, μ_0 are the electric permittivity and the magnetic permeability of free space. We can derive the equation of motion of the electric field $\boldsymbol{E}$ by decoupling the magnetic field $\boldsymbol{B}$ from Eq.(2.3). We assume that the wave travels in a vacuum. Using Eq. (2.3), we write

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{d}{dt}(\boldsymbol{\nabla} \times \boldsymbol{B}). \tag{2.5}$$

By using the vector identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} = -\nabla^2 \boldsymbol{E} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E})$ and Eq. (2.4) it is straightforward to derive the wave equation of the electric field $\boldsymbol{E}$ as

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{d^2 \boldsymbol{E}}{dt^2} = 0, \tag{2.6}$$

where $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ is the speed of light in a vacuum. Similarly, the wave equation of the magnetic field $\boldsymbol{B}$ is given by

$$\nabla^2 \boldsymbol{B} - \frac{1}{c^2}\frac{d^2 \boldsymbol{B}}{dt^2} = 0. \tag{2.7}$$

2.1.1 Vector Potential

The magnetic field in Eq.(2.2) can be written in terms of a vector potential $\boldsymbol{A}$ as

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{2.8}$$

If we substitute Eq.(2.8) into Eq.(2.3), we obtain

$$\boldsymbol{\nabla} \times \big(\boldsymbol{E} + \frac{d\boldsymbol{A}}{dt}\big) = 0. \tag{2.9}$$

Now, by considering an arbitrary scalar function Φ and using this relation $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\Phi = 0$ [23, p. 121]. In detials, the gradient of the scalar function $\nabla\Phi = (\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z})$, therefore $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\Phi = (\frac{\partial\Phi}{\partial yz} - \frac{\partial\Phi}{\partial zy}, \frac{\partial\Phi}{\partial zx} - \frac{\partial\Phi}{\partial xz}, \frac{\partial\Phi}{\partial xy} - \frac{\partial\Phi}{\partial yx}) = 0$. With the help of Eq.(2.9) we obtain

$$\boldsymbol{E} = -\frac{d\boldsymbol{A}}{dt} - \nabla\Phi. \tag{2.10}$$

By substituting Eq.(2.8) and Eq.(2.10) into Eq.(2.4), we get

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = -\frac{1}{c^2}\frac{d}{dt}(\frac{d}{dt}\boldsymbol{A} + \nabla\Phi). \tag{2.11}$$

Using the vector identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = -\nabla^2\boldsymbol{A} + \nabla(\boldsymbol{\nabla} \cdot \boldsymbol{A})$, the previous equation can be reduced to the following form

$$-\nabla^2\boldsymbol{A} + \frac{1}{c^2}\frac{d^2\boldsymbol{A}}{dt^2} = \nabla(\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{d\Phi}{dt}). \tag{2.12}$$

Now, gauge transformation plays an important role in reducing Eq.(2.12) to a simple format. Choosing Lorentz gauge, i.e., $\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{d\Phi}{dt} = 0$, transforms Eq.(2.12) to

$$-\nabla^2\boldsymbol{A} + \frac{1}{c^2}\frac{d^2\boldsymbol{A}}{dt^2} = 0. \tag{2.13}$$

The other type of a gauge transformation is the Coulomb gauge $\boldsymbol{\nabla}.\boldsymbol{A} = 0$, using Eq.(2.1) and Eq.(2.10) one obtains

$$-\frac{d\boldsymbol{\nabla}.\boldsymbol{A}}{dt} - \nabla^2\Phi = 0 \tag{2.14}$$

The expression above can be reduced to

$$\nabla^2 \Phi = 0 \tag{2.15}$$

2.2 Classical Hamiltonian of an Electric Field

The vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ can be written as the sum of all possible modes in a cavity.

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_k \sum_\lambda \boldsymbol{A_{k\lambda}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)}. \tag{2.16}$$

From Eq.(2.8) and Eq.(2.10) we can write the electric and magnetic fields as

$$\boldsymbol{E}(\boldsymbol{r}, t) = \sum_k \sum_\lambda \boldsymbol{e}_{k,\lambda} E_{k,\lambda}(\boldsymbol{r}, t), \tag{2.17}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \sum_k \sum_\lambda \frac{\boldsymbol{k} \times \boldsymbol{e}_{k,\lambda}}{k} B_{k,\lambda}(\boldsymbol{r}, t), \tag{2.18}$$

where the amplitudes in the previous equations are related by

$$E_{k,\lambda}(r, t) = i\omega_k \big[A_{k,\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)} + \mathbf{c.c}\big] \tag{2.19}$$

$$B_{k,\lambda}(r, t) = \frac{i}{c}\omega_k \big[A_{k,\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_k t)} + \mathbf{c.c}\big]. \tag{2.20}$$

The amplitude A_k can be written in terms of canonical variables p and q as

$$A_k = (\frac{1}{2\omega_k \epsilon_0 V})^{\frac{1}{2}} [\omega_k q_k + i p_k], \tag{2.21}$$

$$A_k^* = (\frac{1}{2\omega_k \epsilon_0 V})^{\frac{1}{2}} [\omega_k q_k - i p_k]. \tag{2.22}$$

The classical Hamiltonian or the energy of the electromagnetic field is given by

$$H = \frac{1}{2}\int_v [\epsilon_0 |\boldsymbol{E}(r,t)|^2 + \frac{1}{\mu_0}|\boldsymbol{B}(r,t)|^2]\mathbf{dv}. \tag{2.23}$$

The Hamiltonian we can eventually become [24, p. 22]

$$H = \sum_k \frac{1}{2}(p_k^2 + \omega_k q_k^2) \tag{2.24}$$

where ω is the angular frequency.

2.3 Quantum Treatment of the Electric Field

In the late 19th century, some phenomena were successfully explained classically, such as the photoelectric effect [25, p. 20]. In contrast, other effects could not be dealt with in a classical way, such as the result of the Michelson-Morley Interferometer experiment. Plank proposed to quantize the energy using the harmonic oscillator as a model without quantizing the electric field $\boldsymbol{E}$. In the previous section, we got the Hamiltonian of the classical field and we write Hamiltonian using the canonical variables p and q [26]. In the next section, we introduce the concept of operators in quantum mechanics in order to proceed to the quantization of the electric field.

2.4 Operators

Moving from the classical treatment to the quantum treatment, we need to introduce the concept of operators. Operators in quantum mechanics play a significant role, in which an operator $\hat{A}$ acts, for instance, on an arbitrary state vector $|a\rangle$ and then transforms it to another state $|b\rangle$ [27, p.16]. Any quantum system with a dynamical variable such as momentum or energy gives information about the quantum system [28, p.39]. These variables are known as observables. Therefore, each dynamical variable has a corresponding operator.

2.5 Quantization of the Electric Field

We start with the simple cavity that is subjected to a field. This field can be a single-mode field or multimode fields. Following the traditional way of introducing the canonical variables q and p and then replacing them with quantum operators, each variable has a corresponding operator $q \to \hat{q}$ and $p \to \hat{p}$. Therefore, the vector potential in Eq.2.16 transfers to

$$A_{k\lambda}(\boldsymbol{r}, t) = \sqrt{\frac{1}{2\omega_k \epsilon_0 V}} \left[\omega_k \hat{q}_{k\lambda} + i\hat{p}_{k\lambda}\right] \tag{2.25}$$

$$A^*_{k\lambda}(\boldsymbol{r}, t) = \sqrt{\frac{1}{2\omega_k \epsilon_0 V}} \left[\omega_k \hat{q}_{k\lambda} - i\hat{p}_{k\lambda}\right] \tag{2.26}$$

Operators should satisfy commutation relations

$$[\hat{q}_{k\lambda}, \hat{p}_{k'\lambda'}] = i\hbar \delta_{kk'} \delta_{\lambda\lambda'} \tag{2.27}$$

$$[\hat{q}_{k\lambda}, \hat{q}_{k'\lambda'}] = [\hat{p}_{k\lambda}, \hat{p}_{k'\lambda'}] = 0 \tag{2.28}$$

In addition, these operators should satisfy the canonical commutation relations

$$[\hat{q}, \hat{p}] = i\hbar \hat{I}, \tag{2.29}$$

where $\hat{I}$ is the identity operator. Now, let's define the operators that represent the photon in quantum mechanics.

These are the annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$ [29, p.132]

$$\hat{a}_{k\lambda} = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{q}_{k\lambda} + i\hat{p}_{k\lambda}), \tag{2.30}$$

$$\hat{a}^\dagger_{k\lambda} = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{q}_{k\lambda} - i\hat{p}_{k\lambda}) \tag{2.31}$$

where $\hat{q}_{k\lambda}$ and $\hat{p}_{k\lambda}$ can be written in term of $\hat{a}^\dagger$ and $\hat{a}$ as

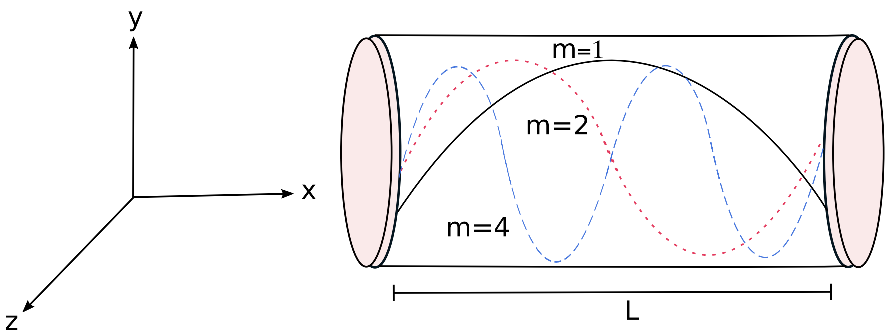

Figure 2.1: A cavity of length L supports many different cavity modes.

$$\hat{q}_{k\lambda} = \sqrt{\frac{\hbar}{2\omega_k}}(\hat{a}^\dagger_{k\lambda} + \hat{a}_{k\lambda}), \tag{2.32}$$

$$\hat{p} = i\sqrt{\frac{\hbar\omega_k}{2}}(\hat{a}^\dagger_{k\lambda} - \hat{a}_{)}. \tag{2.33}$$

A first step to quantize the electric and magnetic fields is to quantize the amplitude $A_{k,\lambda}$.

$$A_{k,\lambda}(\boldsymbol{r}, t) = \sqrt{\frac{\hbar}{2\omega_k \epsilon_0 V}}\left[\hat{a}_{k,\lambda} e^{i(k.r - \omega_k t)} + H.C\right]. \tag{2.34}$$

The electric field can be found using $\boldsymbol{E} = -\frac{d\boldsymbol{A}}{dt}$ and the magnetic field can be found using

$\boldsymbol{B} = (\boldsymbol{\nabla} \times \boldsymbol{A})$. The total electric and magnetic fields can be expressed as

$$\hat{E}(\boldsymbol{r}, t) = \hat{E}^{+}(\boldsymbol{r}, t) + \hat{E}^{-}(\boldsymbol{r}, t), \tag{2.35}$$

$$\hat{B}(\boldsymbol{r}, t) = \hat{B}^{+}(\boldsymbol{r}, t) + \hat{B}^{-}(\boldsymbol{r}, t), \tag{2.36}$$

where

$$\hat{E}^{+}(\boldsymbol{r}, t) = \sum_{k,\lambda} \boldsymbol{e}_{k,\lambda} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \hat{a}_{k,\lambda} e^{-i(\boldsymbol{k}.\boldsymbol{r} - \omega_k t)}, \tag{2.37}$$

$$\hat{E}^{-}(\boldsymbol{r}, t) = \sum_{k,\lambda} \boldsymbol{e}_{k,\lambda} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \hat{a}^{\dagger}_{k,\lambda} e^{i(\boldsymbol{k}.\boldsymbol{r} - \omega_k t)}, \tag{2.38}$$

$$\hat{B}^{+}(\boldsymbol{r}, t) = \sum_{k,\lambda} \boldsymbol{k} \times \boldsymbol{e}_{k,\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} \hat{a}_{k,\lambda} e^{-i(\boldsymbol{k}.\boldsymbol{r} - \omega_k t)}, \tag{2.39}$$

$$\hat{B}^{-}(\boldsymbol{r}, t) = \sum_{k,\lambda} \boldsymbol{k} \times \boldsymbol{e}_{k,\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} \hat{a}^{\dagger}_{k,\lambda} e^{i(\boldsymbol{k}.\boldsymbol{r} - \omega_k t)}. \tag{2.40}$$

After some algebraic manipulation, the Hamiltonian takes this form

$$\hat{H} = \hbar\omega(\hat{a}\hat{a}^{\dagger} + \frac{1}{2}), \tag{2.41}$$

where ω is the frequency of the photon and $\hbar$ is Plank's constant. $\hat{a}$ and $\hat{a}^{\dagger}$ are the annihilation and creation operators of the photon, respectively.

Chapter 3

Treatment of Atom-Field Interaction

3.1 Semi-Classical Treatment of Atom-Field Interaction

In this section, we present the semi-classical treatment of atom-field interaction problem [25]. In this treatment, the field is assumed to be classical, while the atom is treated quantum mechanically. We consider a classical electromagnetic field interacting with a two-level atom as shown in Fig. (3.1).

The quantum state of the atom can be written as

$$\psi(t) = C_e(t)\,|e\rangle + C_g(t)\,|g\rangle\,, \tag{3.1}$$

where $|e\rangle$ is the excited state of the atom with the corresponding probability amplitude C_e(t). $|g\rangle$ is the ground state of the atom, and $C_g(t)$ is the probability amplitude, which

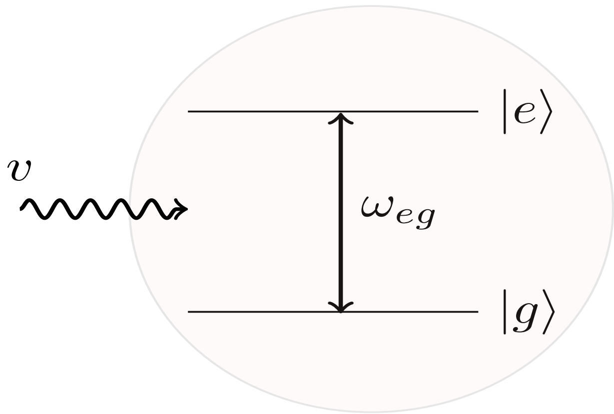

Figure 3.1: A single two-level atom with a transition frequency ω_{eg} interacting with a classical field with frequency v.

corresponds to this state. The system's Hamiltonian can be defined as

$$H_{tot} = H_{free} + H_{int}, \tag{3.2}$$

where H_{free} represents the free part of the Hamiltonian (unperturbed), and H_{int} describes the interaction part of the Hamiltonian. Using the completeness relation $|e\rangle\langle e|+|g\rangle\langle g| = 1$ [30], H_{free} can be written as

$$\begin{aligned} H_{free} &= (|e\rangle\langle e| + |g\rangle\langle g|)H_{free}(|e\rangle\langle e| + |g\rangle\langle g|) \\ &= \hbar\omega_e \, |e\rangle\langle e| + \hbar\omega_g \, |g\rangle\langle g| \, , \end{aligned} \tag{3.3}$$

where $\hbar\omega_e$ and $\hbar\omega_g$ are the energies of the excited state and the ground state of the atom, respectively. Similarly, the interaction Hamiltonian H_{int} can be written using the completeness relation as

$$
\begin{aligned}
H_{int} &= -eyE(t) \\
H_{int} &= -e(|e\rangle\langle e| + |g\rangle\langle g|)y(|e\rangle\langle e| + |g\rangle\langle g|)E(t) \\
&= -e\Big((\rho_{eg} |e\rangle \langle g|) + (\rho_{ge} |g\rangle \langle e|) \Big) E(t),
\end{aligned} \tag{3.4}
$$

where $\rho_{eg} = \langle e| y |g\rangle$ is the dipole moment and y is the linear polarization direction of the electric field. Where the diagonal matrix elements of y vanish $\langle g| y |g\rangle = 0$ same as $\langle e| y |e\rangle = 0$. The applied electric field can takes the form, $E(t) = \varepsilon \cos(\nu t)$ where ν is the frequency of the field and ε is its amplitude. Using the time-dependent Schrödinger equation along with the total Hamiltonian and the wavefunction Eq.(3.1), the equations of motion for the amplitudes C_e and C_g are given by

$$
\frac{dC_e(t)}{dt} = -i\omega_e C_e(t) + i\Omega_R e^{-i\theta} \cos(\nu t) C_g(t), \tag{3.5}
$$

$$
\frac{dC_g(t)}{dt} = -i\omega_e C_g(t) + i\Omega_R e^{i\theta} \cos(\nu t) C_e(t), \tag{3.6}
$$

where $\Omega_R = (\rho_{eg}\varepsilon/\hbar)$ is the Rabi frequency and θ is the phase of the dipole matrix element. In order to solve the coupled differential Eqs.(3.5) and (3.6), we introduce the slowly varying amplitudes [25, p.152]

$$
c_e = C_e e^{i\omega_e t}, \tag{3.7}
$$

$$
c_g = C_g e^{i\omega_g t}. \tag{3.8}
$$

From equations (3.5)-(3.8), it is straightforward to show that the equations of motion for the slowly varying amplitudes are given by the following expressions

$$\begin{aligned}\dot{c}_e &= i\frac{\Omega_R}{2}e^{-i\theta}e^{i(\omega_e-\omega_g)t}c_g(e^{ivt}+e^{-ivt})\\ &= i\frac{\Omega_R}{2}e^{-i\theta}e^{i\omega_{eg}t}c_g(e^{ivt}+e^{-ivt}),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\dot{c}_g &= i\frac{\Omega_R}{2}e^{i\theta}e^{i(\omega_g-\omega_e)t}c_g(e^{ivt}+e^{-ivt})\\ &= i\frac{\Omega_R}{2}e^{i\theta}e^{-i\omega_{eg}t}c_g(e^{ivt}+e^{-ivt}),\end{aligned} \tag{3.10}$$

where $\omega_{eg} = \omega_e - \omega_g$. in the previous equations, we see terms that have oscillating functions of the forms $e^{\pm i(\omega_{eg}+v)t}$ and $e^{\pm i(\omega_{eg}-v)t}$. Here, we can apply the rotating wave approximation [31] where we can neglect the fast oscillating terms. These are the terms that are accompanied by $e^{\pm i(\omega_{eg}+v)t}$. By solving Eqs.(3.9) and (3.10) we get

$$c_e(t) = (a\ e^{i({\Omega_R}^2+(\omega_{eg}-v)^2t)/2} + b\ e^{-i({\Omega_R}^2+(\omega_{eg}-v)^2t)/2})e^{i(\omega_{eg}-v)t/2}, \tag{3.11}$$

$$c_g(t) = (c\ e^{i({\Omega_R}^2+(\omega_{eg}-v)^2)/2} + d\ e^{-i({\Omega_R}^2+(\omega_{eg}-v)^2)/2})e^{-i(\omega_{eg}-v)t/2}, \tag{3.12}$$

where the constants a, b, c, and d can be expressed in terms of the initial conditions of the two-level atom. For simplicity, we define $\Delta = \omega_{eg} - v$ and $\Omega = \sqrt{\Omega_R^2 + (\omega_{eg}-v)^2}$. From Eqs.(3.11) and (3.12), we get

$$\dot{c}_e(0) = a\ \frac{\Omega+\Delta}{2} + b\ \frac{\Omega-\Delta}{2}, \tag{3.13}$$

$$\dot{c}_g(0) = c\ \frac{\Omega-\Delta}{2} - d\ \frac{\Omega+\Delta}{2}. \tag{3.14}$$

Similarly, from Eqs(3.9) and (3.10), we get $\dot{c}_e(0) = \frac{\Omega_R}{2}e^{i\theta}$ and $\dot{c}_g(0) = \frac{\Omega_R}{2}e^{i\theta}$. Therefore, we can express the constants a, b, c, and d as

$$a = \frac{1}{\Omega}[(\Omega - \Delta)c_e(0) + \Omega_R e^{-i\theta} c_g(0)], \tag{3.15}$$

$$b = \frac{1}{\Omega}[(\Omega + \Delta)c_e(0) - \Omega_R e^{-i\theta} c_g(0)], \tag{3.16}$$

$$c = \frac{1}{\Omega}[(\Omega + \Delta)c_g(0) + \Omega_R e^{i\theta} c_e(0)], \tag{3.17}$$

$$d = \frac{1}{\Omega}[(\Omega + \Delta)c_g(0) - \Omega_R e^{i\theta} c_e(0)]. \tag{3.18}$$

By substituting the constants a, b, c, and d into Eqs.(3.9) and (3.10), we get

$$c_e(t) = \left[c_e(0)\left[\cos\left(\frac{\Omega t}{2}\right) - \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right] + i\frac{\Omega_R}{\Omega}e^{-i\theta}c_g(0)\sin\left(\frac{\Omega t}{2}\right)\right]e^{i\Delta t/2}, \tag{3.19}$$

$$c_g(t) = \left[c_g(0)\left[\cos\left(\frac{\Omega t}{2}\right) + \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right] + i\frac{\Omega_R}{\Omega}e^{-i\theta}c_e(0)\sin\left(\frac{\Omega t}{2}\right)\right]e^{-i\Delta t/2}. \tag{3.20}$$

From the previous solutions of the probability amplitudes $c_e(t)$ and $c_g(t)$, we can calculate the probability of finding the atom in the ground state or the excited state. If the atom is initially in the excited state, i.e., $c_e(0) = 1$ and $c_g(0) = 0$, then the probability of finding the atom in the excited state when, i.e, $\Delta = 0$

$$P_e(t) = |c_e(t)|^2 = \cos^2\left(\frac{\Omega t}{2}\right). \tag{3.21}$$

Another quantity we can find is the population inversion, which is given by

$$\begin{aligned} W(t) &= |c_e(t)|^2 - |c_g(t)|^2, \\ W(t) &= \cos^2(\frac{\Omega t}{2}) - \sin^2(\frac{\Omega t}{2}). \end{aligned} \tag{3.22}$$

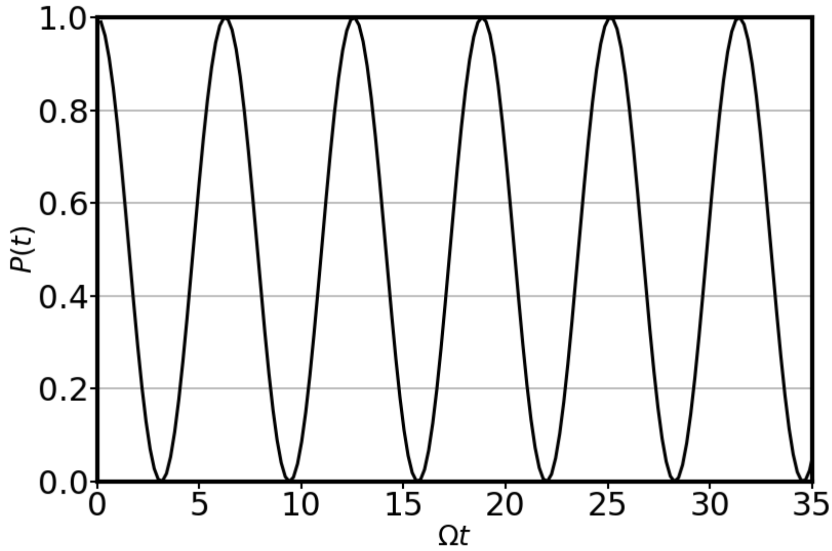

Figure 3.2: Excitation probability of a single two-level atom initially prepared in the excited state $|e\rangle$ with $\Delta = 0$.

At resonance, $\Delta = 0$ and $\Omega = \Omega_R$, therefore the inversion reduces to the following expression

$$W(t) = cos(\Omega_R t). \tag{3.23}$$

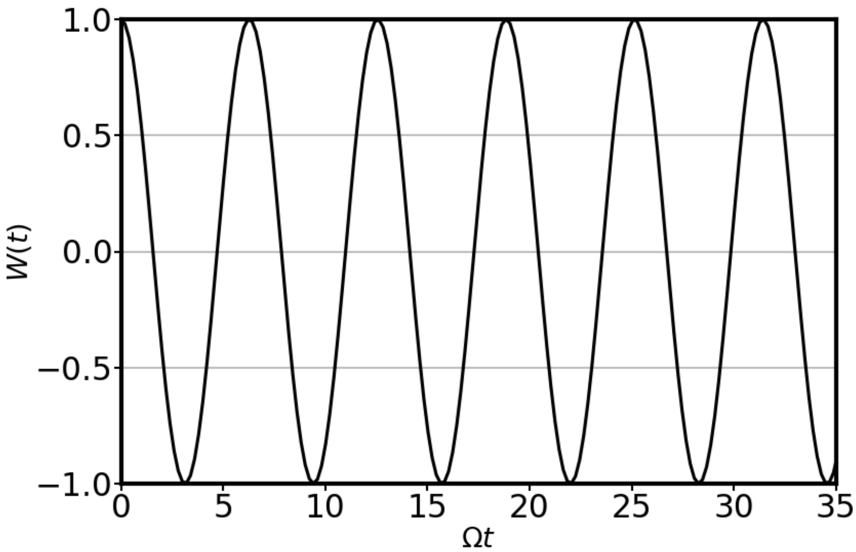

Figure 3.3: Inversion of a single two-level atom initially prepared in the excited state $|e\rangle$ with $\Delta = 0$. The inversion oscillates between -1 and 1.

3.2 Quantum Treatment of Atom-field Interaction

In this section, we present the quantum treatment of the interaction between two-level atom and a single mode field [32].

We start by writing the Hamiltonian of the total system in the dipole approximation as

$$H_{tot} = H_{atom} + H_{field} - e\boldsymbol{r}.\boldsymbol{E}, \tag{3.24}$$

where $H_{atom} = \hbar\omega_e |e\rangle\langle e| + \hbar\omega_g |g\rangle\langle g|$ is the Hamiltonian of the atom with ω_e and ω_g representing the energies of the excited state and the ground state of the atom, respectively. The Hamiltonian of the single mode quantized field is given as $H_{field} = \hbar v\hat{a}^\dagger\hat{a}$ with v being the frequency of the field. Last term in Eq. (3.24) describes the interaction energy operator between the atom and the field. Using the basis of the two-level atom, the atomic dipole moment $e\boldsymbol{r}$ can be rewritten as $e\boldsymbol{r} = \sum_{ij} e\,|i\rangle\,\langle i|\, r\,|j\rangle\,\langle j| = e(\rho_{eg}\sigma_+ + \rho_{ge}\sigma_-)$. Where $\rho_{ge} = e\,\langle g|\, r\,|e\rangle$ is the electric dipole matrix, also $\sigma_- = |g\rangle\,\langle e|$ and $\sigma_+ = |e\rangle\,\langle g|$. The single mode quantized field can be expressed as $\boldsymbol{E} = \hat{\epsilon}\varepsilon(\hat{a} + \hat{a}^\dagger)$, where $\hat{\epsilon}$ is a unit vector and ε is an arbitrary amplitude of the field. Therefore, the total Hamiltonian can be written as

$$H_{tot} = \hbar\omega_e |e\rangle\langle e| + \hbar\omega_g |g\rangle\langle g| + \hbar v \hat{a}^\dagger \hat{a} + \hbar g(\sigma_- + \sigma_+)(\hat{a} + \hat{a}^\dagger), \tag{3.25}$$

where $g = -\frac{\vec{\rho}_{ge}.\hat{\epsilon}\varepsilon}{\hbar}$ represent the coupling strength between the atom and the field. The unit vector $\hat{\epsilon}$ and the polarization which is a linear polarization are taken in consideration to simplify the calculation. The sketch in Fig.3.4 represents the energy levels of the system.

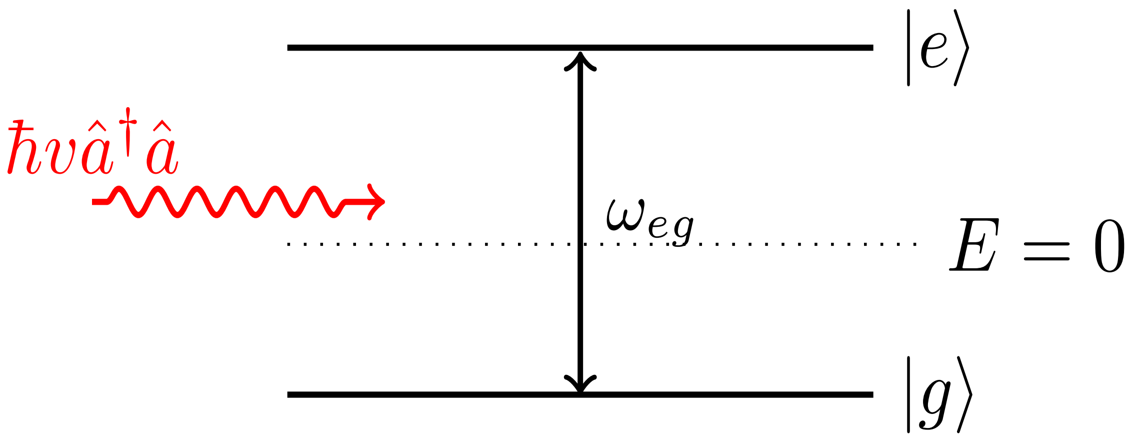

Figure 3.4: Two-level atom interacting with a single mode quantized field.

Using the fact that the atomic Hamiltonian can be written as

$$H_{atom} = \frac{1}{2}\hbar\omega_{eg}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{\hbar}{2}(\omega_g + \omega_e), \tag{3.26}$$

therefore, we can express the total Hamiltonian to become

$$H_{tot} = \frac{\hbar}{2}\omega_{eg}\sigma_z + \hbar v \hat{a}^\dagger \hat{a} + \hbar g(\sigma_- + \sigma_+)(\hat{a} + \hat{a}^\dagger), \tag{3.27}$$

where $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ and $\omega_{eg} = \omega_e - \omega_g$. The third term of the total Hamiltonian Eq. (3.27) contains the following four terms

- $\hat{a}\,|e\rangle\langle g|$ —> $\hat{a}\sigma_+$,
- $\hat{a}\,|g\rangle\langle e|$ —> $\hat{a}\sigma_-$,
- $\hat{a}^\dagger\,|e\rangle\langle g|$ —> $\hat{a}^\dagger\sigma_+$,
- $\hat{a}^\dagger\,|g\rangle\langle e|$ —> $\hat{a}^\dagger\sigma_-$,

where $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$. By omitting the non-energy conserving terms, i.e., $\hat{a}\sigma_-$ and $\hat{a}^\dagger\sigma_+$. The first term is $\hat{a}\sigma_-$ in which the atom takes transition from the excited state to the ground state and lost energy as an absorption of photon. The other term, $\hat{a}^\dagger\sigma_+$, the atom takes transition from the lower state to the excited state and emits photon the total Hamiltonian becomes.

$$H_{tot} = \frac{1}{2}\hbar\omega_{eg}\sigma_z + \frac{1}{2}\hbar v\hat{a}^\dagger\hat{a} + \hbar g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-). \tag{3.28}$$

The previous form of the Hamiltonian can be used to describe the interaction between a two-level atom and a single mode cavity field. The total quantum state describing a single two-level atom placed inside a cavity field can be written as

$$\psi(t) = \sum_{n=0}^{\infty} c_{e,n}(t)\,|e, n\rangle + c_{g,n+1}(t)\,|g, n+1\rangle\,, \tag{3.29}$$

where $|e, n\rangle$ means that the atom is in the ground state and the cavity contains n photon, while the state $|g, n+1\rangle$ describes the case when the atom goes to the ground state resulting in having $n+1$ photon in the cavity. $c_{e,n}(t)$ and $c_{g,n+1}(t)$ are the associated probability amplitudes for the system to be in the states $|e, n\rangle$ and $|g, n+1\rangle$, respectively. Now, we introduce the interaction picture, which transfers the Hamiltonian to a different form

$$\mathcal{H} = e^{iH_{af}t/\hbar} H_{int} e^{-iH_{af}t/\hbar}, \tag{3.30}$$

where $H_{af} = \frac{1}{2}\hbar\omega_{eg}\sigma_z + \frac{1}{2}\hbar v a^\dagger a$ and $H_{int} = \hbar g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-)$. Using the Baker-Hausdorff lemma [33, p.13], we get

$$\mathcal{H} = \hbar g\big(\hat{a}\sigma_+ e^{i(\omega_{eg}-v)t} + \hat{a}^\dagger\sigma_- e^{-i(\omega_{eg}-v)t}\big). \tag{3.31}$$

Using the time-dependent Schrödinger equation, we can derive the equations of motions for the probability amplitudes using the Hamiltonian Eq. (3.31) along with the state of the total system Eq. (3.29). These equations of motions are given by

$$\dot{c}_{e,n} = -ig\sqrt{n+1}c_{g,n+1}(t)e^{i(\omega_{eg}-v)t}, \tag{3.32}$$

$$\dot{c}_{g,n+1} = -ig\sqrt{n+1}c_{e,n}(t)e^{-i(\omega_{eg}-v)t}. \tag{3.33}$$

Initially, we can assume that the cavity contains n photons and the atom is in the excited state, therefore the solutions of the previous equations of motions are

$$c_{e,n}(t) = \cos\Big(g\sqrt{n+1}t\Big), \tag{3.34}$$

$$c_{g,n+1}(t) = -i\sin\Big(g\sqrt{n+1}t\Big). \tag{3.35}$$

3.3 Weisskopf-Wigner Approximation

In the previous section, the two-level atom is assumed to be isolated inside the cavity and not subject to any damping process. In this section, we consider that the two-level atom is placed in the vacuum and can interact with the vacuum modes. The Hamiltonian of this system can be written as

$$H_{tot} = \frac{1}{2}\hbar\omega_{eg}\sigma_z + \frac{1}{2}\sum_k \hbar v\hat{a_k}^\dagger\hat{a_k} + \sum_k \hbar g(r)\left(\hat{a}_k\sigma_+e^{i(\omega_{eg}-v_k)t} + \hat{a}_k^\dagger\sigma_-e^{-i(\omega_{eg}-v_k)t}\right), \quad (3.36)$$

where we sum over k because the atom is subjected to an arbitrary vacuum mode. As the coupling strength between the atom and the vacuum modes is spatially dependent. It can be written as [25] $g_k(r) = g_k e^{-ik.r}$. If the atom is initially set to be in the exited state and the field contains no photons, then the time-dependent state of the system can be written as

$$\psi(t) = C_e(t)\hat{a}\left|e,1\right\rangle + \sum_k C_{gk}(t)\hat{a_k}^\dagger\left|g,0_k\right\rangle, \quad (3.37)$$

where $C_e(t)$ and $C_g(t)$ are probability amplitudes. The equations of motion for $C_e(t)$ and $C_g(t)$ can be derived using the time-dependent Schrödinger equation as mention in the previous section. The equations of motion are given by

$$\dot{C}_e(t) = \sum_k -ig^\star e^{ikr}e^{i(\omega_{eg}-v_k)t}C_g(t), \quad (3.38)$$

$$\dot{C}_g(t) = \sum_k -ige^{ikr}e^{-i(\omega_{eg}-v_k)t}C_e(t). \quad (3.39)$$

By integrating Eq.(3.39) and substituting it back into Eq.(3.38), we get

$$C_g(t) - C_g(0) = -ige^{ikr}\int_0^t e^{-i(\omega_{eg}-v_k)t'}C_e(t')dt', \quad (3.40)$$

where $C_g(0) = 0$ from the initial conditions. By substituting Eq.(3.40) into Eq.(3.38), we get

$$\dot{C}_e(t) = -\sum_k |g|^2 \int_0^t C_e(t^{'}) e^{i(\omega_{eg} - v_k)(t - t^{'})} dt^{'}. \tag{3.41}$$

If we consider that all frequencies of the modes are very close, we can replace the sum in Eq.(3.41) by an integral i.e., the $\sum_k$ changes to $\frac{2V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta sin\theta \int_0^{\infty} dkk^2$, therefore, the resultant expression for the equation of motion of $C_e(t)$ is

$$\dot{C}_e(t) = -\frac{\mu_{ge}^2}{6\pi^2 c^3 \hbar \epsilon_0} \int_0^{\infty} v_k^3 dv_k \int_0^t C_e(t^{'}) e^{i(\omega_{eg} - v_k)(t - t^{'})} dt^{'}. \tag{3.42}$$

where the coupling strength is defined as $|g_k|^2 = \frac{v_k \rho_{ge}^2}{2\hbar\epsilon_0 V} cos^2\theta$. According to the Weisskopf–Wigner approximation [34], the time-dependent integral in Eq. (3.42) can be divided into three regions:

- $t^{'} << t$ the fast oscillation.
- $t^{'} >> t$ there is no contribution to the integral.
- $t^{'} \approx t$ then we can change $C_e(t^{'})$ to $C_e(t)$.

As $C_e(t^{'})$ can be replaced by $C_e(t)$ around $t^{'} \approx t$, Eq. (3.42) becomes

$$\dot{C}_e(t) = -\frac{\mu_{ge}^2}{6\pi^2 c^3 \hbar \epsilon_0} \int_0^{\infty} v_k^3 dv_k \int_0^t C_e(t) e^{i(\omega_{eg} - v_k)(t - t^{'})} dt^{'}. \tag{3.43}$$

Because of the small change of v_k^3 in the frequency regime $v_k \approx \omega_{eg}$, this allows us to replace v_k by ω_{eg} in Eq. (3.43), and to also change the integration limit to start from

$-\infty$ to ∞. Therefore, we get the following form for the equation of motion of $C_e(t)$

$$\dot{C}_e(t) = -\frac{\omega_{eg}^3 \mu_{ge}^2}{6\pi^2 c^3 \hbar \epsilon_0} \int_{-\infty}^{\infty} dv_k \int_0^t C_e(t) e^{i(\omega_{eg}-v_k)(t-t')} dt', \tag{3.44}$$

where the previous integrals can be solved analytically. Solving the integrals in the previous equations gives

$$\dot{C}_e(t) = -\frac{\omega_{eg}^3 \mu_{ge}^2}{3\pi \epsilon_0 \hbar c^3} C_e(t). \tag{3.45}$$

It is straightforward to show that

$$C_e(t) = exp\big(-\frac{\Gamma}{2}t\big). \tag{3.46}$$

where $\Gamma = \frac{\omega_{eg}^3 \mu_{ge}^2}{3\pi \epsilon_0 \hbar c^3}$ is the decay rate. The probability to find the atom in the excited state is given by

$$|C_e(t)|^2 = e^{-\Gamma t}, \tag{3.47}$$

which indicate that the two-level atom exponentially decays with rate Γ as shown in Fig.(3.5).

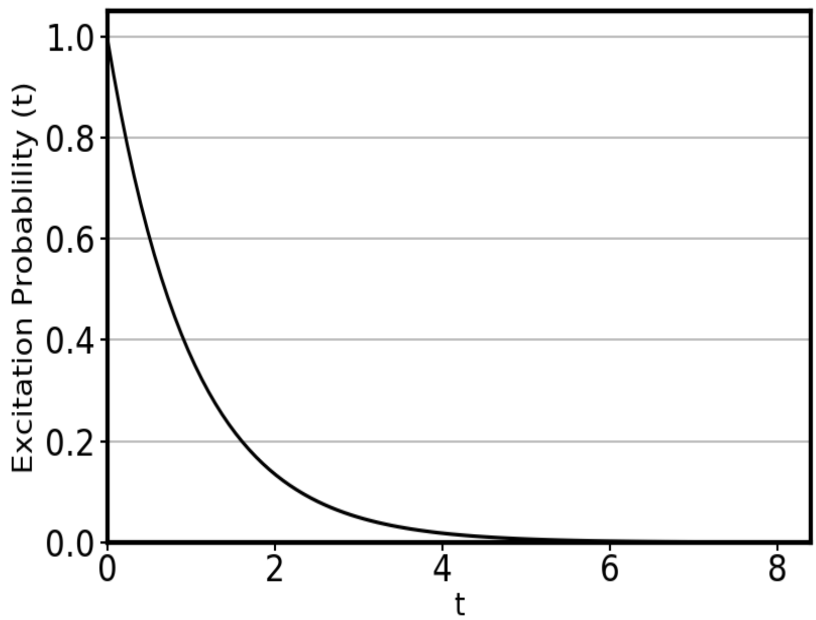

Figure 3.5: Excitation probability of a single two-level atom initially prepared in the excited state $|e\rangle$ and placed in vacuum.

Chapter 4

Propagation of a Single-Photon Pulse in Two Parallel Waveguides Coupled to a Two-Level Atom

4.1 Linearization of Dispersion

The dispersion of a photonic waveguide [35] is shown in Fig. (4.1), and can be described by the following equation

$$\omega_k = \pm\sqrt{c^2k^2 + \omega_{cf}^2}, \tag{4.1}$$

where ω_{cf} is a cutoff frequency.

$$\omega_{k\approx k_0}(k) = \omega_0(k_0) + (\frac{d\omega}{dk})_{k\approx k_0}\delta k + (\frac{d^2\omega}{dk^2})_{k\approx k_0}\delta k^2 + ..., \tag{4.2}$$

As the focus of this thesis is studying the propagation of a narrow-bandwidth photon pulse in a single-mode waveguide, the nonlinear part in Eq.(4.2) are not of interest.

If we choose a frequency ω_0 that is away from the cutoff frequency, then we see that around the corresponding wave numbers $\pm k_0$, the dispersion can be considered linear [35] as shown in Fig. (4.1) i.e.,

$$\omega_k \approx \omega_0 + (|k| - k_0)v_g, \tag{4.3}$$

where v_g is the group velocity and k_0 is the wave number which corresponds to the frequency ω_0.

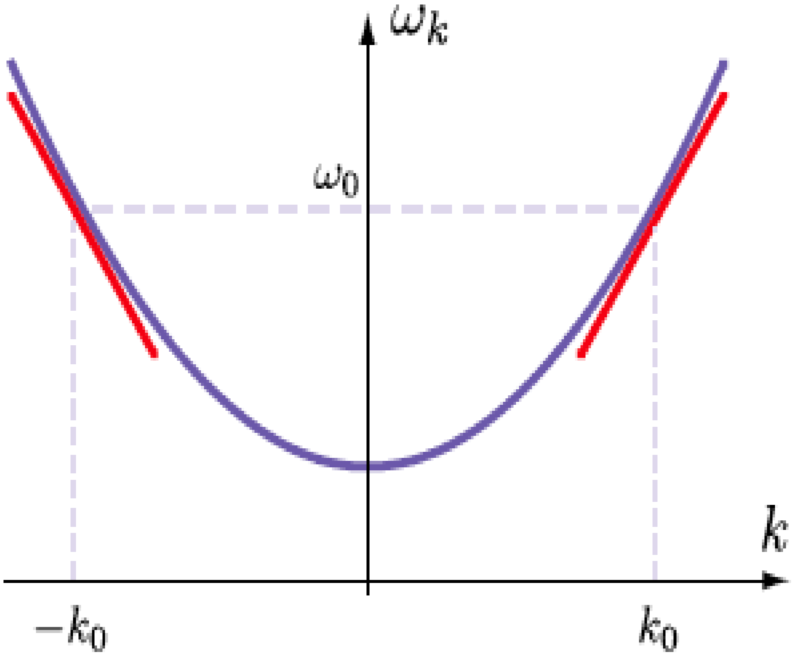

Figure 4.1: Dispersion of a single mode photonic waveguide. Around $\pm k_0$, the dispersion can is considered to be linear. ω_0 is far away from the cutoff frequency. (courtesy of APS [35])

It is worth mentioning that frequencies that are below the cutoff frequency will not propagate through the waveguide, whereas frequencies that are above the cutoff frequency will travel through the waveguide.

4.2 Previous Study

Before we proceed to the problem of a single photon transport in a system containing a two-level atom coupled to two parallel waveguides, we first present the previous study [13, 36] about the propagation of a photon pulse in a waveguide coupled to a single two-level atom. The system studied in [13, 36] is sketched in Fig. (5.2).

The total Hamiltonian of this system is given by

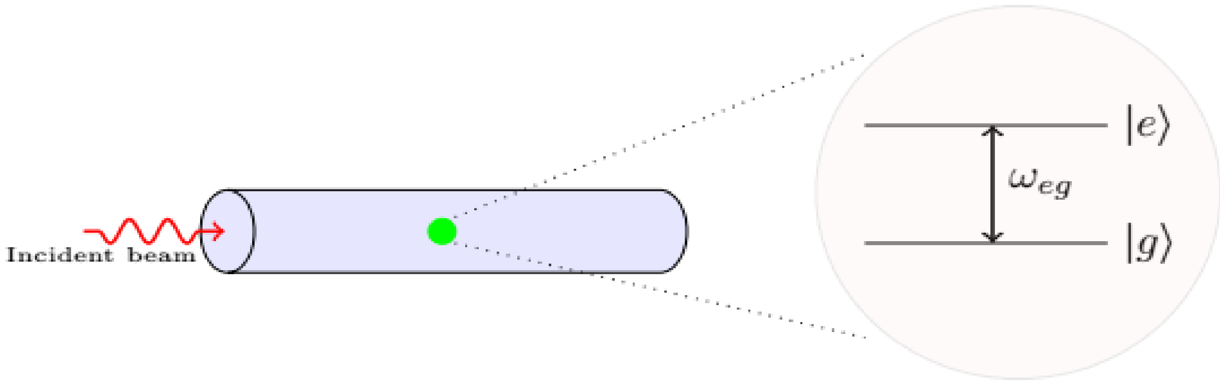

Figure 4.2: Propagation of a photon pulse through a 1-D waveguide coupled to a single two level atom

$$H_{tot} = H_{free} + H_{int}, \tag{4.4}$$

Where H_{free} accounts for the free part of the Hamiltonian, while H_{int} is the interaction Hamiltonian. The free Hamiltonian is given by

$$H_{free} = \hbar\omega_{eg} |e\rangle\langle e| + \sum_k \hbar\omega_k a_k^\dagger a_k, \tag{4.5}$$

where ω_{eg} is the atomic transition frequency of the two-level atom and ω_k represents the frequency of the waveguide mode. It is assumed that the ground state sets to be zero energy. $a_k^\dagger(a_k)$ is the creation (annihilation) operator of the guided photon. The interaction Hamiltonian is given as

$$\hat{H}_{int} = \sum_k \hbar \left[g_k e^{ikr} a_k |e\rangle\langle g| + g_k^* e^{-ikr} a_k^\dagger |g\rangle\langle e| \right], \tag{4.6}$$

where g_k is the coupling strength between the two-level atom and the guided photon, and

r is the position of the atom. The total state of the system can be written at any time as

$$|\psi(t)\rangle = C_e(t)e^{-i\omega_{eg}t}|e,0\rangle + \sum_k e^{-i\omega_k t}C_{g,k}(t)|g,1_k\rangle, \tag{4.7}$$

where the amplitude $C_e(t)$ gives the excitation probability of the atom. Using the time-dependent Schrödinger equation, we derive the equations of motion for the amplitudes $C_e(t)$ and $C_{g,k}(t)$. These equations of motion are given by

$$\dot{C_e}(t) = -i\sum_k C_{g,k}(t)e^{ikr}g_k e^{-i\delta\omega_k t}, \tag{4.8}$$

$$\dot{C_{g,k}}(t) = -iC_e(t)e^{-ikr}g_k^* e^{i\delta\omega_k t}, \tag{4.9}$$

where $\delta\omega_k = \omega_k - \omega_{eg}$. By integrating Eq.(4.9) from 0 to t one can obtain

$$C_{g,k}(t) = C_{g,k}(0) - i\int_0^t C_e(t^{'})e^{i\delta\omega_k t^{'}}dt^{'}, \tag{4.10}$$

since the atom is initially prepared in the ground state, the single-photon pulse has an amplitude $C_{g,k}(0) = \frac{(8\pi)^{\frac{1}{4}}}{\sqrt{\Delta L}}e^{-\frac{(k_{eg}+\delta k+k_0)^2}{\Delta^2}}$ and by integrating Eq. (4.8) and substituting it into Eq. (4.7), we get

$$\dot{C_e}(t) = -i\sum_k g_k e^{ikr}C_{g,k}(0)e^{-i\delta\omega_k t} - \sum_k |g_k|^2 e^{-ikr}\int_0^t C_e(t^{'})e^{i\delta\omega_k t^{'}}e^{-i\delta\omega_k t}dt^{'}. \tag{4.11}$$

The sum over k in Eq. (4.11) can be changed to an integral if we consider that the waveguide is long, i.e.,$\sum_k \to \frac{L}{2\pi}\int_{-\infty}^{\infty}dk$. The previous equation can be solved using Weisskopf-Wigner approximation as explained in the previous chapter and therefore one can obtain

$$\dot{C}_e(t) = b(t) - \frac{\Gamma}{2}C_e(t), \tag{4.12}$$

where $b(t) = \frac{-i}{2\pi}\sqrt{\frac{\Gamma v_g L}{2}}\int_{-\infty}^{\infty} C_g(0)e^{ikr}e^{-i\delta w_k t}dk$. The solution to Eq. (4.12) can be expressed as it follows [36]

$$C_e(t) = -iS\Big[\operatorname{erf}(D + \sqrt{B}t) - \operatorname{erf}(D)\Big]e^{\frac{-\Gamma}{2}t}, \tag{4.13}$$

where $D = (A - 2Bt_0)/2\sqrt{B}$, $t_0 = \frac{r}{v_g}$, $A = \frac{-\Gamma}{2} + i(k_0 - k_{eg})v_g$, $B = \frac{\Delta^2 v_g^2}{4}$, $S = (\frac{\pi}{8})^{\frac{1}{4}}\sqrt{\frac{\Gamma}{\Delta v_g}}e^{-At_0+A^2/4B}e^{ik_a r}$, erf(x)= $2/\sqrt{\pi}\int_0^x e^{-t^2}dt$, and Δ is the width of Gaussian beam in the k space. The long time ($t \longrightarrow \infty$) spectra of the right and left propagating photons can be found by the Fourier transformations as [36]

$$C_g^R(t \to \infty) = -\frac{2i\delta k v_g C_{g,\delta k}(0)}{1 - 2i\delta k v_g/\Gamma}, \tag{4.14}$$

$$C_g^L(t \to \infty) = -e^{2(k_a + i\delta k)r}\frac{C_{g,\delta k}(0)}{1 - 2i\delta k v_g/\Gamma}, \tag{4.15}$$

where $C_{g,\delta k}(0)$ is the initial photon pulse amplitude. The probability for the atom to be in the excited state is shown in Fig. 4.3 using Eq.(4.13). The atom is initially in the ground state. It can be seen in Fig. 4.3 that as the photon pulse excites the atom starting at $t \approx 7.5$, and the maximum excitation occurs at $t = 11$.

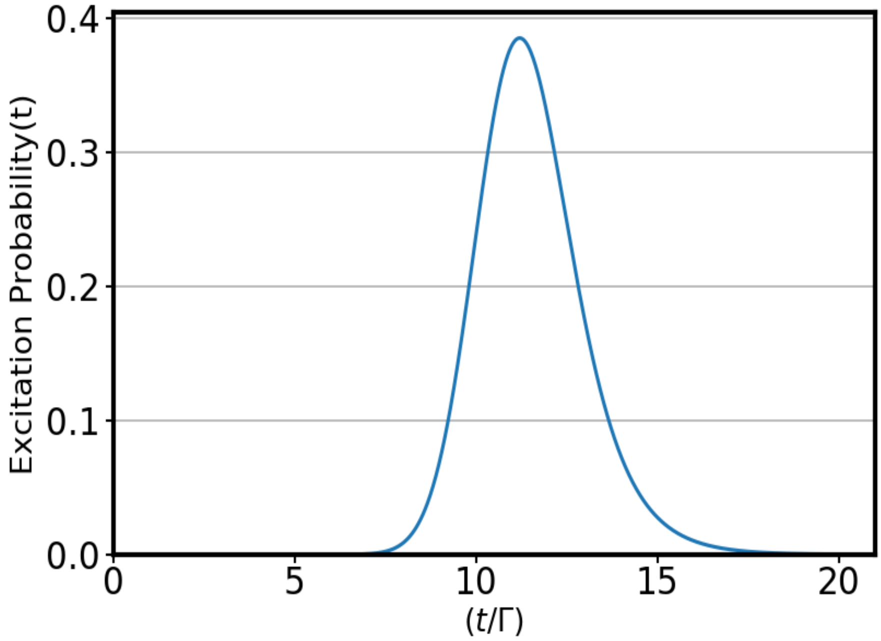

Figure 4.3: Excitation probability of a two-level atom placed in a 1-D waveguide. The atom is at the origin $r = 0$, and the photon pulse is initially $10/\Delta$ away from the atom. Where the width of Gaussian beam $\Delta = 1$ and the $v_g = 1$.

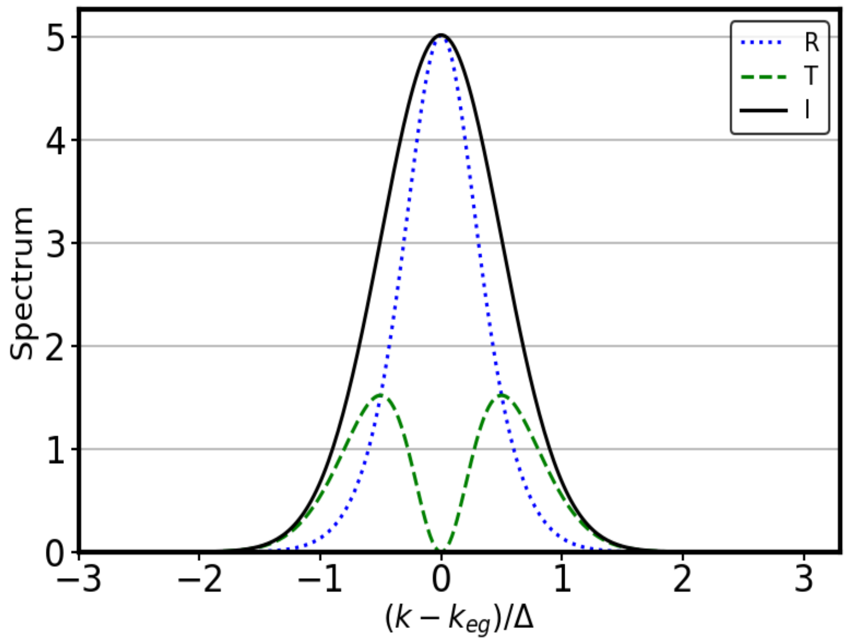

Figure 4.4: Spectra of the incident (solid), reflected (dotted), and transmitted (dashed) photon pulse. At resonance, the pulse is completely reflected.

From Eqs.(4.14) and (4.15), we can obtain the spectra of the reflected and transmitted photons. The spectra of the incident, reflected, and transmitted photons are plotted in Fig. 4.4. At resonance, i.e., $k = k_{eg}$, a complete reflection of the photon pulse occurs. In Fig. 4.5, we plot the pulse shape (intensity) of the incident, reflected, and transmitted photons with respect to the position of the two-level atom, which is placed at $r = 0$. The reflected pulse maintains the same shape as the incident pulse but with smaller intensity, while the transmitted pulse shows the result of the interference between the incident and re-emitted photons [36].

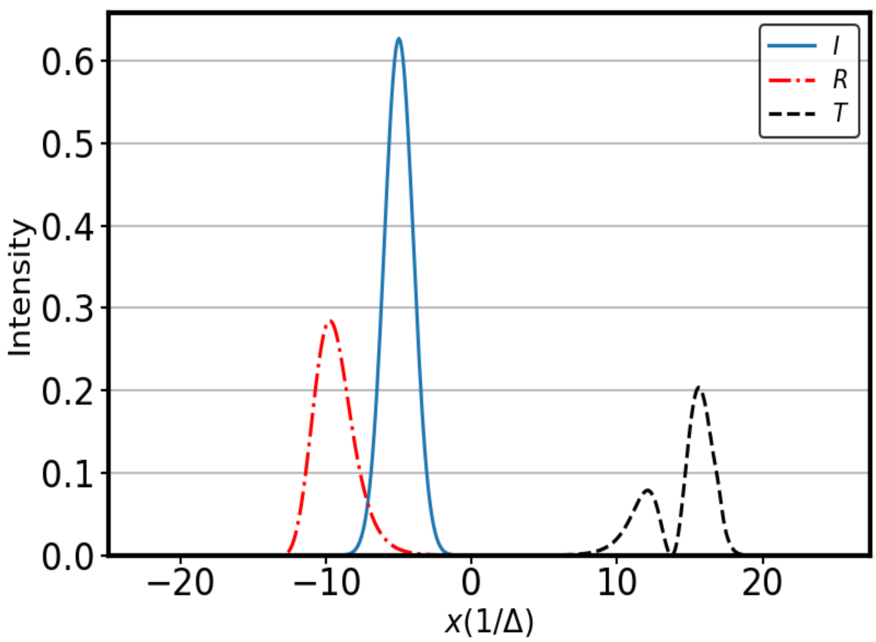

Figure 4.5: Pulse shape of the incident (solid) at $t = \frac{6}{\Delta}$, reflected (dotted-dashed) at $t = \frac{15}{\Delta}$, and transmitted (dashed) photon pulse at $t = \frac{15}{\Delta}$. The two-level atom is at $x = 0$.

4.3 Two-Level Atom Coupled to Two Parallel Waveguides

4.3.1 Description of the System

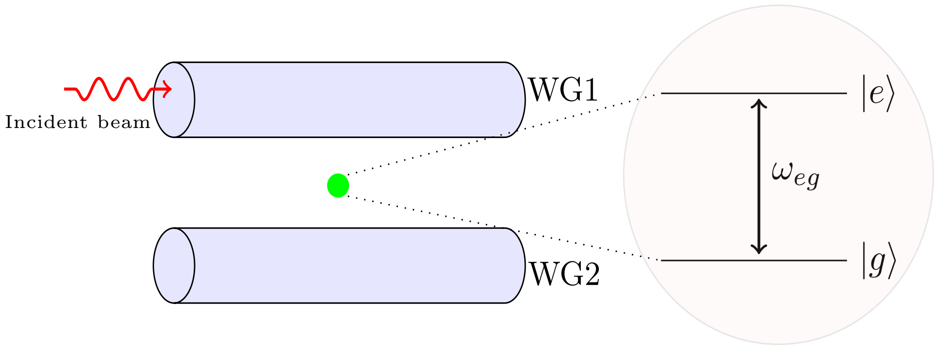

Figure 4.6: A single two-level atom coupled to two parallel waveguides.

In this section, we consider a different system than the one presented in the previous section. As shown in Fig. (4.6), we consider a two-level atom placed between two parallel waveguides and coupled to each of them with the same coupling strength. The atom consists of a ground state $|g\rangle$ with a frequency ω_g and an excited state $|e\rangle$ with a frequency ω_e. We consider a single photon pulse propagating from left in the upper waveguide (WG1). We study the real time evolution of this system and compare the results to the system presented in the previous section.

4.3.2 Hamiltonian of The System

The total Hamiltonian of the system can be written as

$$\hat{H}_{tot} = \hat{H}_{atom} + \hat{H}_w + \hat{H}_{int}, \tag{4.16}$$

where $\hat{H}_{atom}$ and $\hat{H}_w$ are the atom and waveguides Hamiltonian, whereas $\hat{H}_{int}$ represents the interaction Hamiltonian between the atom and the two waveguides. The atomic Hamiltonian can be written as

$$\hat{H}_{atom} = \hbar\omega_{eg} |e\rangle \langle e| , \tag{4.17}$$

where ω_{eg} represents the frequency difference between the two atomic levels $\omega_{eg} = \omega_e - \omega_g$. The Hamiltonian of the two waveguides can be expressed as

$$\hat{H}_\omega = \sum_{k_1} \hbar\omega_{k_1} \hat{a}^\dagger{}_{k1} \hat{a}_{k1} + \sum_{k_2} \hbar\omega_{k_2} \hat{a}^\dagger{}_{k2} \hat{a}_{k2}, \tag{4.18}$$

where k_1 and k_2 are the modes of the first and second waveguide, respectively. ω_{k_1} and ω_{k_2} are the frequencies of the guided photons in the first and second waveguides, respectively. $\hat{a}^\dagger_{k_1}(\hat{a}_{k_1})$ is the creation (annihilation) operator of the guided photon in the second waveguide (WG1), and $\hat{a}_{k_2}(\hat{a}_{k_2})$ is the creation (annihilation) operator of the guided photon in the first waveguide (WG2). The interaction Hamiltonian can be written as

$$\begin{aligned} \hat{H}_{int} = & \sum_{k_1} \hbar g_{k_1} e^{ikr} \hat{a}_{k1} |e\rangle \langle g| + \hbar g^*_{k1} e^{-ikr} \hat{a}^\dagger_{k1} |g\rangle \langle e| \\ & + \sum_{k_2} \hbar g_{k_2} e^{ikr} \hat{a}_{k2} |e\rangle \langle g| + \hbar g^*_{k_2} e^{-ikr} \hat{a}^\dagger_{k2} |g\rangle \langle e| , \end{aligned} \tag{4.19}$$

where r represents the position of the atom and is set to be $r = 0$ for simplicity. g_{k_1} represents the coupling strength between the atom and the first waveguide (WG1), and g_{k_2} is the coupling strength between the atom and the second waveguide (WG2).

The time-dependent state of the system can be written as

$$|\psi(t)\rangle = \alpha_1(t)e^{-i\omega_{eg}t}\,|e,0_{k_1},0_{k_2}\rangle + \sum_{k_1}\beta_1(t)e^{-i\omega_{k_1}t}\,|g,1_{k_1},0_{k_2}\rangle + \sum_{k_2}\gamma_1(t)e^{-i\omega_{k_2}t}\,|g,0_{k_1},1_{k_2}\rangle\,, \tag{4.20}$$

where $|e, 0_{k_1}, 0_{k_2}\rangle$ describes the state of the system in which the atom is in the excited state, and there is no photon present in either the first waveguide (WG1) or the second waveguide (WG2). However, $|g, 1_{k_1}, 0_{k_2}\rangle$ is state of the system where the atom is in the ground state and there is one photon in the first waveguide (WG1), and no photon is in the second waveguide (WG2). The state $|g, 0_{k_1}, 1_{k_2}\rangle$ describes the case in which the atom is in the ground state and there is only a photon present in the second waveguide (WG2). Using time-dependent Schrödinger equation, the equations of motion for the probability amplitudes can be expressed as

$$\dot{\alpha_1}(t) = -i\sum_{k_1} g_{k_1}\beta_1(t)e^{-ikr}e^{-\delta\omega_{k_1}t} - i\sum_{k_2} g_{k_2}\gamma_1(t)e^{ikr}e^{-\delta\omega_{k_2}t}, \tag{4.21a}$$

$$\dot{\beta_1}(t) = -i\alpha_1(t)\sum_{k_1} g^*_{k_1}e^{-ikr}e^{i\delta\omega_{k_1}t}, \tag{4.21b}$$

$$\dot{\gamma_1}(t) = -i\alpha_1(t)\sum_{k_2} g^*_{k_2}e^{-ikr}e^{i\delta\omega_{k_2}t}, \tag{4.21c}$$

where $\delta\omega_{k_1} = \omega_{k_1} - \omega_{eg}$ and $\delta\omega_{k_2} = \omega_{k_2} - \omega_{eg}$. Now, by integrating equations (4-21b) and (5-19c) and then inserting them back into equation (4-21a), we get

$$\begin{aligned}\dot{\alpha_1}(t) = {} & -i\sum_{k_1} g_{k_1}\beta_1(0)e^{ikr}e^{i\delta\omega_{k_1}t} - \sum_{k_1}\left|g^2_{k_1}\right|\int_0^t \alpha_1(t')e^{i\delta\omega_{k_1}(t'-t)}dt' \\ & - \gamma_1(0)\sum_{k_2} g_{k_2}e^{ikr}e^{i\delta\omega_{k_2}t} - \sum_{k_2}\left|g^2_{k_2}\right|\int_0^t \alpha_1(t')e^{i\delta\omega_{k_2}(t'-t)}dt',\end{aligned} \tag{4.22}$$

The atom initially prepare to be in the ground state. Therefore, $\beta_1(0)$ is the initial photon pulse, which is incident upon the first waveguide (WG1). $\beta_1(0)$ is our Gaussian beam in this case. $\gamma_1(0) = 0$ means that there is no input pulse in the second waveguide (WG2). $g_{k_1} = \sqrt{\frac{v_g\Gamma_1}{2L_1}}$ and $g_{k_2} = \sqrt{\frac{v_g\Gamma_2}{2L_2}}$, are the coupling strengths between the atom and the (WG1) and (WG2), respectively. If we consider that both waveguides are very long, then the sums over k_1 and k_2 can be converted to integrals. Therefore, Eq.(4.22) can be written as

$$\begin{aligned}\dot{\alpha_1}(t) = \frac{-i}{2\pi}\sqrt{\frac{v_g L_1 \Gamma_1}{2}}\int_{-\infty}^{\infty}\beta_1(0)e^{ikr}e^{-i\delta\omega_{k_1}t}dk_{k_1} - \frac{L_1}{2\pi}|g_{k_1}^2|\int_{-\infty}^{\infty}dk_{k_1}\int_0^t \alpha_1(t')e^{i\delta\omega_{k_1}(t'-t)}dt' \\ -\frac{-i}{2\pi}\sqrt{\frac{v_g\Gamma_2 L_2}{2}}\int_{-\infty}^{\infty}\gamma_1(0)e^{ikr}e^{-i\delta\omega_{k_2}t}dk_{k_2} - \frac{L_2}{2\pi}|g_{k_2}^2|\int_{-\infty}^{\infty}dk_{k_2}\int_0^t \alpha_1(t')e^{i\delta\omega_{k_2}(t'-t)}dt',\end{aligned} \tag{4.23}$$

where we consider $L_1 = L_2 = L$ for simplicity. The integrals in the second and last term in the previous equation can be solved using Weisskopf Wigner approximation. The resultant expression for the equation of motion of the excitation probability reduces to

$$\begin{aligned}\dot{\alpha_1}(t) = \frac{-i}{2\pi}\sqrt{\frac{v_g\Gamma_1 L}{2}}\int_{-\infty}^{\infty}\beta_1(0)e^{ikr}e^{i\delta\omega_{k_1}t}dk_{k_1} - \frac{i}{2\pi}\sqrt{\frac{v_g\Gamma_2 L}{2}}\int_{-\infty}^{\infty}\gamma_1(0)e^{ikr}e^{i\delta\omega_{k_2}t}dk_{k_2} \\ -\frac{(\Gamma_1+\Gamma_2)}{2}\alpha_1(t),\end{aligned} \tag{4.24}$$

where $\Gamma_1 = \frac{2L|g_1|^2}{v_g}$ and $\Gamma_2 = \frac{2L|g_2|^2}{v_g}$. By writing $a_1(t) = \frac{-i}{2\pi}\sqrt{\frac{v_g L}{2}}\int_{-\infty}^{\infty}\beta_1(0)e^{ikr}e^{i\delta\omega_{k_1}t}dk_{k_1}$ and $a_2(t) = -\frac{i}{2\pi}\sqrt{\frac{v_g L}{2}}\int_{-\infty}^{\infty}\gamma_1(0)e^{ikr}e^{i\delta\omega_{k_2}t}dk_{k_2}$, then Eq. (4.24) becomes

$$\dot{\alpha}(t) = a_1(t) + a_2(t) - \frac{\Gamma_1}{2}\alpha(t) - \frac{\Gamma_2}{2}\alpha(t). \tag{4.25}$$

As we only consider the incoming pulse propagating to the first waveguide, then $a_2(t) = 0$ as $\gamma_1(0) = 0$. We consider a Gaussian shape pulse of the form $\beta_1(0) = \frac{(8\pi)^{\frac{1}{4}}}{\sqrt{\Delta L}}e^{-\frac{(k_{eg}+\delta k-k_0)^2}{\Delta^2}}$

[37, 38] propagating into WG1, where $\delta k = k - k_{eg}$, k_{eg} is the wave number corresponding to the atomic transition frequency ω_{eg}, and k_0 is the central wave number of the Gaussian pulse. The solution of the differential equation (4.25) gives the excitation probability of the two-level atom, which is coupled to the two parallel waveguides. The photon pulse travels inside the first waveguide (WG1) in which the atom is placed at the origin, i.e., $r = 0$. We choose that $k_0 = k_{eg}$. At the initial time $t = 0$ the photon pulse is $(\frac{10}{\Gamma})$ away from the two-level atom. When comparing the result of the excitation probability shown in Fig. 4.7 to the result of the excitation probability of the two-level atom, which is coupled to a 1-D waveguide [36] Fig. 4.3, it is clear that the atom can reach a larger probability of excitation than the two-level atom coupled to the two parallel waveguides.

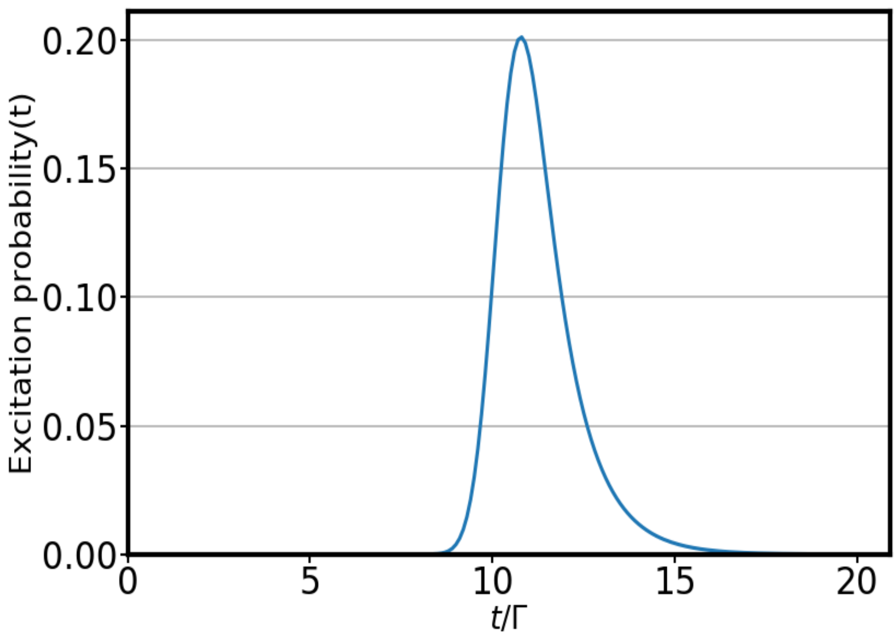

Figure 4.7: Excitation probability of the two-atom.

4.3.3 Spectrum

We follow the same method presented in [37] to find the photon spectrum at the long time limit ($t \longrightarrow \infty$). From Eq. (4-19b), we can write the left and right propagating modes in the first waveguide (WG1) as

$$\begin{aligned} \beta_1^R(t) &= \beta_1(0) - i\sqrt{\frac{\Gamma_1 v_g}{2L}} e^{-i(k_{eg}+\delta k)r} \int_0^t \alpha_1(t^{'}) e^{i\delta k_1 v_g t^{'}} dt^{'}, \\ \beta_1^L(t) &= -i\sqrt{\frac{\Gamma_1 v_g}{2L}} e^{i(k_{eg}+\delta k)r} \int_0^t \alpha_1(t^{'}) e^{i\delta k_1 v_g t^{'}} dt^{'}. \end{aligned} \tag{4.26}$$

We now define the following equation

$$\chi_1(\delta k_1) = \int_{-\infty}^{\infty} \alpha_1(t) e^{i\delta k_1 v_g t} dt \tag{4.27}$$

At the long time limit ($t \longrightarrow \infty$) changes the interval of integration from (0,t) into $(-\infty, \infty)$ where $\alpha_1(t) = 0$ when $t < 0$. By taking Fourier transformation for this equation and putting the results back into Eq. (4.25), we get

$$-i\chi_1(\delta k_1)\delta k_1 v_g = a_1(\delta k_1) - \frac{(\Gamma_1 + \Gamma_2)}{2}\chi_1(\delta k_1). \tag{4.28}$$

Solving the last equation for $\chi_1(\delta k_1)$, we get

$$\chi_1(\delta k_1) = \frac{a_1(\delta k_1)}{\left[\frac{\left(\Gamma_1+\Gamma_2\right)}{2} - i\delta k_1 v_g\right]}. \tag{4.29}$$

Using this expression, we can write the spectra of the right and left propagating modes in WG1 as

$$\beta^R_{\delta k_1}(t \longrightarrow \infty) = \beta_{\delta k_1}(0) - i\sqrt{\frac{\Gamma_1 v_g}{2L}} e^{-i(\delta k_1 + k_{eg})r} \chi_1(\delta k_1), \tag{4.30a}$$

$$\beta^L_{\delta k_1}(t \longrightarrow \infty) = -i\sqrt{\frac{\Gamma v_g}{2L}} e^{i(\delta k_1 + k_{eg})r} \chi_1(\delta k_1), \tag{4.30b}$$

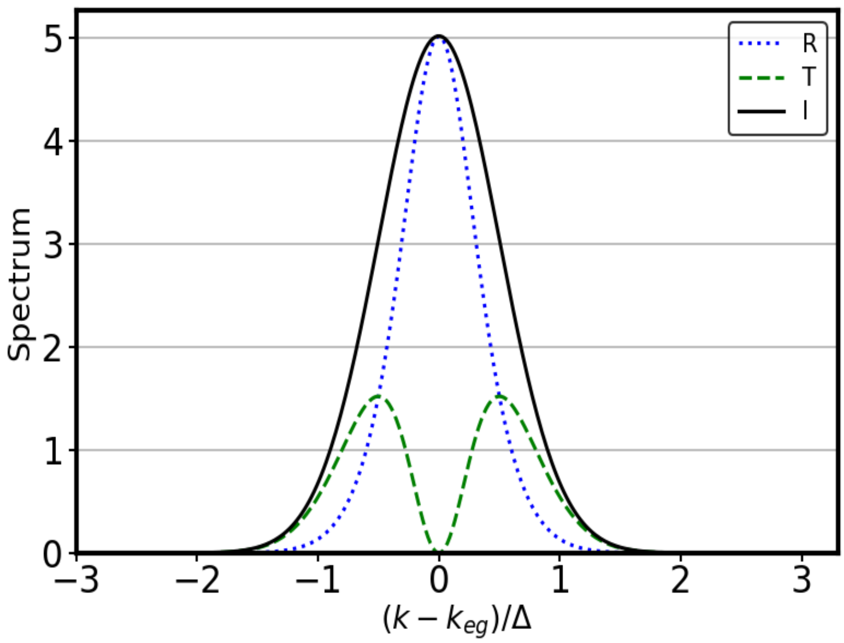

Figure 4.8: Spectra of the incident, reflected, and transmitted photon pulse.

The spectra of the incident, reflected, and transmitted photons are shown in Fig 4.8. These results agree with the findings of the case of a two-level atom coupled to a waveguide in Ref. [36]. Adding the second waveguide (WG2) does not change spectra in Fig 4.8.

4.3.4 Pulse Shape

Using Fourier transformations, we can calculate the pulse shape of the photon modes by the following equations

$$\beta_{x1}^{R}(t) = e^{ik_{eg}x} \int_{-\infty}^{\infty} \beta_{\delta k_1}^{R} e^{i\delta k_1 (x - v_g t)} d\delta k_1, \tag{4.31a}$$

$$\beta_{x1}^{L}(t) = e^{-ik_{eg}x} \int_{-\infty}^{\infty} \beta_{\delta k_1}^{L} e^{-i\delta k_1 (x - v_g t)} d\delta k_1, \tag{4.31b}$$

where $|\beta_{x1}^{R}(t)|^2$ and $|\beta_{x1}^{L}(t)|^2$ give the pulse shape of the right and left propagating photons, respectively. The pulse shape of the incident, reflected, and transmitted photons are plotted in Fig. 4.9. The atom is considered to be at $r = 0$. The peak of the transmission signal is located at $x \approx 16$ and a small dip in the transmission signal is located at $x \approx 14$. The dip in the transmission signal occurs because of interference between the incoming photon and remitted photon. In the case of two-level atom coupled to a 1D waveguide, the transmission has a larger intensity in comparison to the current case. We see that 47% of the incident light is reflected, whereas 25% of the incident light is transmitted.

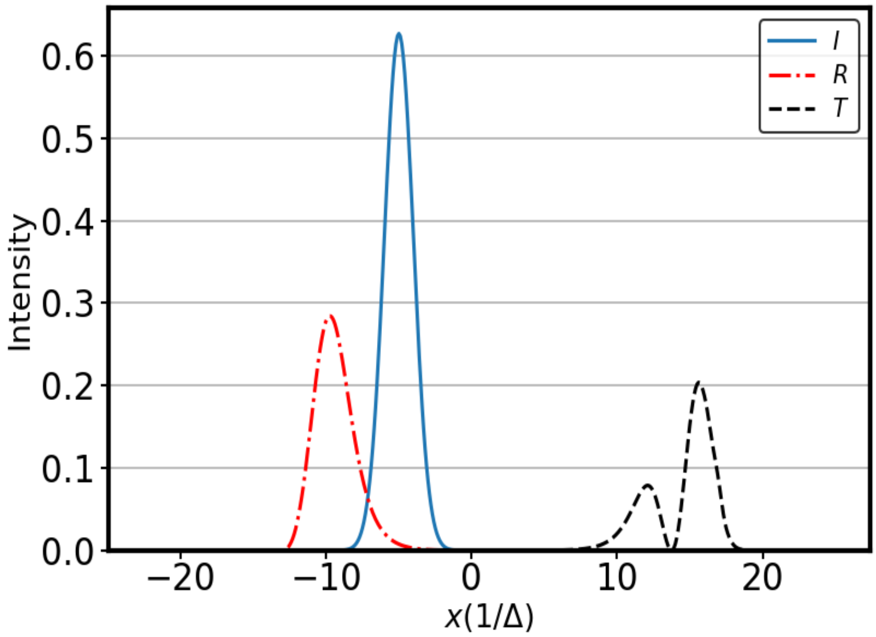

Figure 4.9: Pulse shape of the incident at $t = \frac{6}{\Delta}$, reflected at $t = \frac{15}{\Delta}$, and transmitted photon at $t = \frac{15}{\Delta}$ where the coupling strength $g = 1$.

Chapter 5

Propagation of a Single-Photon Pulse in a 1-D Waveguide Coupled to a Three-Level Atomic System

5.1 Introduction

An incident photon pulse on a 1-D waveguide containing a two-level atom can be completely reflected or transmitted by controlling the frequency of the guided photon and the coupling strength between the atom and the waveguide.

In fact, the idea of using a three-level atom in a 1-D waveguide has been considered [21, 39, 40]. Here, we use a V-type three-level atom in a 1-D waveguide and solve the dynamics of the system. We show that the reflection and transmission of an incident photon pulse on the wavegide can be controlled by an external coherent field.

5.2 The Model

The propagation of a single-photon pulse through a V-type three-level atom coupled to a single mode 1-D waveguide is shown in Fig. (5.1). The atomic states $|e_1\rangle$ and $|g\rangle$ are coupled to the waveguide photon with frequency ω_k, where k corresponds to the wave number of the guided photon. Levels $|e_2\rangle$ and $|e1\rangle$ are coupled by a coherent driving field with Rabi frequency Ω_R.

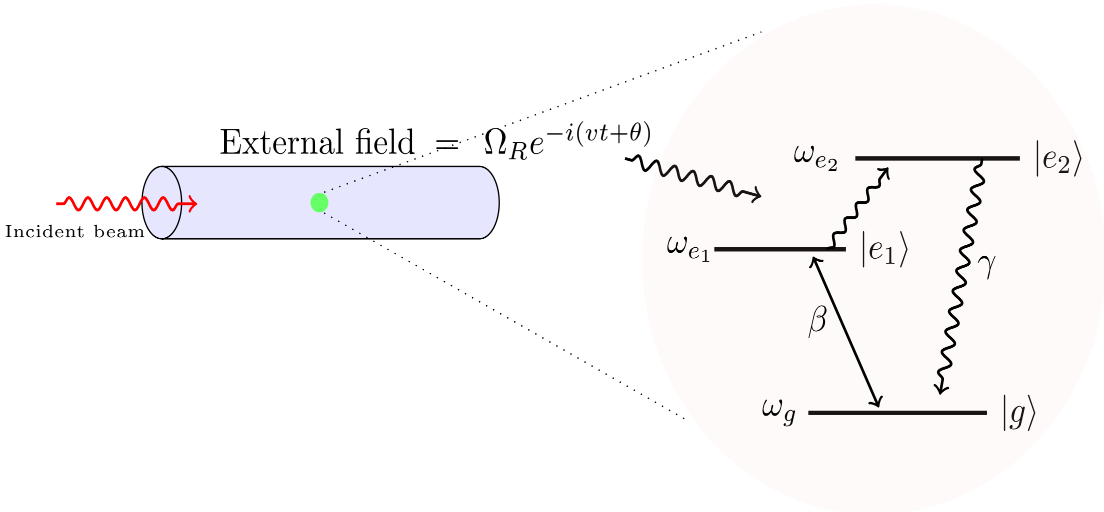

Figure 5.1: A single three-level atom coupled to a 1-D waveguide.

The Hamiltonian of the system can be written as [21]

$$H = H_0 + H_I, \tag{5.1}$$

where H_0 and H_I describe the free and interaction parts of the system's Hamiltonian, respectively. The free part of the Hamiltonian is given by

$$H_0 = \hbar\omega_{e_1} |e_1\rangle \langle e_1| + \hbar\omega_{e_2} |e_2\rangle \langle e_2| + \hbar \sum_k \omega_k (a_k^\dagger a_k + b_k^\dagger b_k), \tag{5.2}$$

where ω_{e_1} and ω_{e_2} correspond to the energies of the states $|e_1\rangle$ and $|e_2\rangle$, respectively.

The creation and annihilation operators for the β photon are $a^\dagger$ and a, while $b^\dagger$ (b) is the creation (annihilation) operator of the γ photon. The interaction Hamiltonian of the system is given by

$$
\begin{aligned}
H_I =& \hbar \sum_k g\Big(e^{-ikr} a_k^\dagger \left|g\right\rangle \left\langle e_1\right| + e^{ikr} a_k \left|e_1\right\rangle \left\langle g\right| \Big) + \hbar \sum_k g\Big(e^{-ikr} b_k^\dagger \left|g\right\rangle \left\langle e_2\right| + e^{ikr} b_k \left|e_2\right\rangle \left\langle g\right| \Big) \\
&+ \hbar\Big(\Omega_R e^{i(\nu t+\phi)} \left|e_1\right\rangle \left\langle e_2\right| + \Omega_R e^{-i(\nu t+\phi)} \left|e_2\right\rangle \left\langle e_1\right| \Big),
\end{aligned}
\tag{5.3}
$$

where the first term in the previous equation describes the interaction between the incident photon pulse and the transition $|e_1\rangle \leftrightarrow |g\rangle$, while the second term describes the interaction energy between the states $|e_2\rangle$ and $|g\rangle$. The coupling strength between the three-level emitter and the waveguide is g, while r is the position of the emitter. The third term in Eq(5.3) corresponds to the interaction between the external driving field and the transition $|e_2\rangle \leftrightarrow |e_1\rangle$ where ν is the frequency of the field with a phase ϕ. The total state of the system at any time can be written as

$$
\begin{aligned}
\psi(t) =& \alpha^{e_1}(t) e^{-i\omega_{e_1} t} \left|e_1, 0_\beta, 0_\gamma\right\rangle + \alpha^{e_2}(t) e^{-i\omega_{e_2} t} \left|e_2, 0_\beta, 0_\gamma\right\rangle + \sum_k \beta_k(t) e^{-i\omega_k t} \left|g, 1_\beta, 0_\gamma\right\rangle \\
&+ \sum_k \gamma_k(t) e^{-i\omega_k t} \left|g, 0_\beta, 1_\gamma\right\rangle,
\end{aligned}
\tag{5.4}
$$

where the state $|e_1, 0_\beta, 0_\gamma\rangle$ ($|e_2, 0_\beta, 0_\gamma\rangle$) indicates that the atom is in the excited state $|e_1\rangle(|e_2\rangle)$ with no photons in the waveguide. However, the state $|g, 1_\beta, 0_\gamma\rangle(|g, 0_\beta, 1_\gamma\rangle)$ corresponds to the case where the atom is the ground state $|g\rangle$ with β (γ) photon in the waveguide. For simplicity, we assume the energy difference between the states $|e_1\rangle$ and $|e_2\rangle$ is resonant with the external field. Using the time-dependent Schrödinger equation, the equations of motion for the probability amplitudes are given by

$$\dot{\alpha}^{e_1}(t) = - i\Omega_R e^{i\theta}\alpha^{e_2}(t) - i\sum_k g^* e^{ikr}\beta(t)e^{i(\omega_{e1}-\omega_k)t}, \quad (5.5a)$$

$$\dot{\alpha_k}^{e_2}(t) = - i\Omega_R e^{-i\theta}\alpha^{e_1}(t) - i\sum_k g e^{ikr}\gamma(t)e^{i(\omega_{e2}-\omega_k)t}, \quad (5.5b)$$

$$\dot{\beta}_k(t) = - ige^{-ikr}\alpha^{e_1}(t)e^{-i(\omega_{e1}-\omega_k)t}, \quad (5.5c)$$

$$\dot{\gamma_k}(t) = - ige^{-ikr}\alpha^{e_2}(t)e^{-i(\omega_{e2}-\omega_k)t}. \quad (5.5d)$$

By integrating Eqs. 5.5(c) and 5.5(d), we get

$$\beta_k(t) = \beta_k(0) - ige^{-ikr}\int_0^t dt' \alpha^{e_1}(t')e^{-i(\omega_{e_1}-\omega_k)t'}, \quad (5.6a)$$

$$\gamma_k(t) = - ige^{-ikr}\int_0^t dt' \alpha^{e_2}(t')e^{-i(\omega_{e_2}-\omega_k)t'}, \quad (5.6b)$$

where $\beta_k(0)$ is the initial amplitude of the incident photon pulse. We consider that the initial photon pulse is a Gaussian pulse, i.e., $\beta_k(0) = \frac{(8\pi)^{1/4}}{\sqrt{\Delta L}}e^{(k_{e_1}+\delta k-k_0)^2/\Delta^2}$, where $\delta k = |k| - k_{e_1}$. By substituting Eq. (6.6) into the equation of motion for the excitation probability Eq. 6.5(a), we get

$$\dot{\alpha}^{e_1}(t) = -i\Omega_R e^{i\theta}\alpha^{e_2}(t) - i\sum_k \left[g\beta(0)e^{ikr}e^{i\delta\omega t} - i|g|^2\int_0^t \alpha^{e_1}(t')e^{i\delta\omega(t-t')}\right], \quad (5.7)$$

where $\delta\omega = \omega_{e_1} - \omega_k$. Using Weisskopf–Wigner approximation, the resultant equation of motion for the amplitude $\alpha^{e_1}(t)$ is

$$\dot{\alpha}^{e_1}(t) = -i(\frac{1}{2\pi})\sqrt{\frac{v_g\Gamma L}{2}}\int_{-\infty}^{\infty}\beta(0)e^{ikr}e^{i(\omega_{e_1}-\omega_k)t}dk - i\Omega_R e^{i\theta}\alpha^{e_2}(t) - \frac{\Gamma}{2}\alpha^{e_1}(t), \quad (5.8)$$

where $g = \sqrt{\frac{v_g \Gamma L}{2}}$ and the value of decay rate $\Gamma = \frac{2L|g|^2}{v_g}$ as found in Eq.(3.45). Similarly, the equation of motion for the amplitude $\alpha^{e_2}(t)$ can be found by following the same procedure above. Therefore, we get

$$\dot{\alpha}^{e_2}(t) = -i\Omega_R e^{-i\theta}\alpha^{e_1}(t) - \frac{\Gamma}{2}\alpha^{e_2}(t). \tag{5.9}$$

Eq. (5.8) and (5.9) can be solved analytically and the solutions for the amplitudes $\alpha^{e_1}(t)$ is given by

$$\begin{aligned} \alpha^{e_1}(t) = \frac{i\pi^{1/4}}{2^{7/4}} e^{A} \Bigg[& e^{B} erf\Big[\frac{-1}{2}\Big(1 + t_0\Gamma + \frac{2i\Omega_R}{\Gamma}\Big)\Big] - e^{B} erf\Big[\frac{1}{2}\Big(-1 + \Gamma t - t_0\Gamma - \frac{2i\Omega_R}{\Gamma}\Big)\Big] \\ & + e^{C} erf\Big[\frac{1}{2}\Big(-1 - t_0\Gamma + \frac{2i\Omega_R}{\Gamma}\Big)\Big] - e^{C} erf\Big[\frac{1}{2}\Big(-1 + \Gamma t - t_0\Gamma + \frac{2i\Omega_R}{\Gamma}\Big)\Big]\Bigg], \end{aligned} \tag{5.10}$$

where

$$A = \frac{(\Gamma - 2i\Omega_R)\Big(\Gamma + 2t_0\Gamma^2 - 2i\Omega_R\Big)}{4\Gamma^2}, \tag{5.11}$$

$$B = \frac{t}{2}\big(-\Gamma - 2i\Omega_R\big) + 2it_0\Omega_R + \frac{2i\Omega_R}{\Gamma}, \tag{5.12}$$

$$C = \frac{t}{2}\big(-\Gamma - 2i\Omega_R\big). \tag{5.13}$$

From the previous equations for $\beta_k(t)$ and $\gamma(t)$, we can write the expressions for the right and left propagating photons as

$$\beta^R_{\delta k}(t) = \beta(0) - i\sqrt{\frac{\Gamma v_g}{2L}} e^{ikr} \int_0^t dt^{'} \alpha^{e_1}(t') e^{-i(\omega_{e_1}-\omega_k)t^{'}}, \tag{5.14a}$$

$$\beta^L_{\delta k}(t) = - i\sqrt{\frac{\Gamma v_g}{2L}} e^{ikr} \int_0^t dt^{'} \alpha^{e_1}(t') e^{-i(\omega_{e_1}-\omega_k)t^{'}}, \tag{5.14b}$$

$$\gamma^R_{\delta k}(t) = - i\sqrt{\frac{\Gamma v_g}{2L}} e^{-ikr} \int_0^t dt^{'} \alpha^{e_2}(t') e^{-i(\omega_{e_2}-\omega_k)t^{'}}, \tag{5.14c}$$

$$\gamma^R_{\delta k}(t) = - i\sqrt{\frac{\Gamma v_g}{2L}} e^{ikr} \int_0^t dt^{'} \alpha^{e_2}(t') e^{-i(\omega_{e_2}-\omega_k)t^{'}}. \tag{5.14d}$$

We follow the same procedure given in [37] to calculate the spectrum of the β and γ photons when $t \to \infty$.

5.3 Results

5.3.1 Spectrum

The spectrum of the right and left propagating β photons when $t \to \infty$ can be calculated from

$$\beta^R_{\delta\omega_k}(t \to \infty) = \beta(0) - i\sqrt{\frac{\Gamma v_g}{2L}} e^{-ikr} \xi_1(\delta\omega_k), \tag{5.15a}$$

$$\beta^L_{\delta\omega_k}(t \to \infty) = - i\sqrt{\frac{\Gamma v_g}{2L}} e^{ikr} \xi_1(\delta\omega_k), \tag{5.15b}$$

where

$$\xi_1(\delta\omega_k) = \int_{-\infty}^{\infty} \alpha^{e_1}(t) e^{i\delta\omega_k t} dt. \tag{5.16}$$

Similarly, the spectrum of the right and left propagating γ photons when $t \to \infty$ can be calculated from

$$\gamma^R(t \to \infty) = -i\sqrt{\frac{\Gamma v_g}{2L}} e^{-ikr} \xi_2(\delta k), \tag{5.17a}$$

$$\gamma^L(t \to \infty) = -i\sqrt{\frac{\Gamma v_g}{2L}} e^{ikr} \xi_2(\delta k), \tag{5.17b}$$

where

$$\xi_2(\delta\omega_k) = \int_{-\infty}^{\infty} \alpha^{e_2}(t) e^{i\delta\omega_k t} dt. \tag{5.18}$$

Making inverse Fourier transformation for Eq. (5.16) and Eq. (5.18) gives

$$\alpha^{e_1}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi_1(\delta\omega_k) e^{-i\delta\omega_k t} d\delta\omega_k, \tag{5.19a}$$

$$\alpha^{e_2}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi_2(\delta\omega_k) e^{-i\delta\omega_k t} d\delta\omega_k. \tag{5.19b}$$

By substituting Eqs. (5.19) into Eq. (5.8) and (5.9), we get the following two linear equations

$$(\frac{\Gamma}{2} - i\delta k v_g)\xi_1(\delta k) + i\Omega_R e^{-i\theta} \xi_2(\delta k) = -i\sqrt{\frac{\Gamma L}{2v_g}} \beta(0), \tag{5.20a}$$

$$(\frac{\Gamma}{2} - i\delta k v_g)\xi_2(\delta k) + i\Omega_R e^{i\theta} \xi_1(\delta k) = 0. \tag{5.20b}$$

By solving the previous equations, we get

$$\xi_1(\delta k) = - i\sqrt{\frac{\Gamma L}{2v_g}}\beta(0)\left[\frac{\frac{\Gamma}{2} - i\delta k v_g}{(\frac{\Gamma}{2} - i\delta k v_g)^2 + \Omega_R^2}\right], \tag{5.21a}$$

$$\xi_2(\delta k) =\sqrt{\frac{\Gamma L}{2v_g}}\beta(0)\left[\frac{\Omega_R e^{i\theta}}{-(\frac{\Gamma}{2} - i\delta k v_g)^2 - \Omega_R^2}\right]. \tag{5.21b}$$

By substituting these two results for $\xi_1(\delta k)$ and $\xi_2(\delta k)$ in Eqs. (5.15) and (5.17), we get the spectrum for the right and left propagating β and γ photons

$$\beta^R =\beta(0)\frac{\Omega_R^2 + (\frac{\Gamma}{2} - i\delta k v_g)^2 - \frac{\Gamma}{2}(\frac{\Gamma}{2} - i\delta k v_g)}{\Omega_R^2 + (\frac{\Gamma}{2} - i\delta k v_g)^2}, \tag{5.22a}$$

$$\beta^L =\beta(0)\frac{-\frac{\Gamma}{2}(\frac{\Gamma}{2} - i\delta k v_g)}{\Omega_R^2 + (\frac{\Gamma}{2} - i\delta k v_g)^2}, \tag{5.22b}$$

$$\gamma^R =i\beta(0)e^{ikr}\left[\frac{\frac{\Gamma}{2}\Omega_R e^{i\theta}}{(\frac{\Gamma}{2} - i\delta k v_g)^2 + \Omega_R^2}\right], \tag{5.22c}$$

$$\gamma^L =i\beta(0)e^{-ikr}\left[\frac{\frac{\Gamma}{2}\Omega_R e^{i\theta}}{(\frac{\Gamma}{2} - i\delta k v_g)^2 + \Omega_R^2}\right]. \tag{5.22d}$$

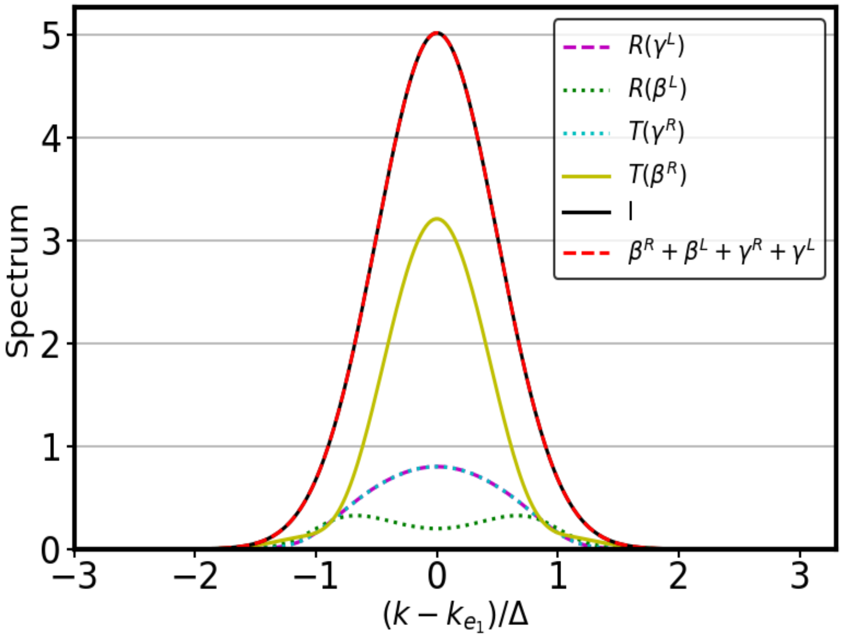

Figure 5.2: Spectra of the incident, reflected β and γ photons, and transmitted β and γ photons. The Rabi frequency $\Omega_R = 1$ and the width of the Gaussian beam $\Delta = 1$.The superscripts R and L represent the right and left propagating modes, respectively.

Fig.5.2 shows the spectra of the reflected and transmitted β and photons when the Rabi frequency $\Omega_R = 1$. At resonance, about 60% of the photon pulse is transmitted, whereas the reflected spectrum is slightly above the zero.

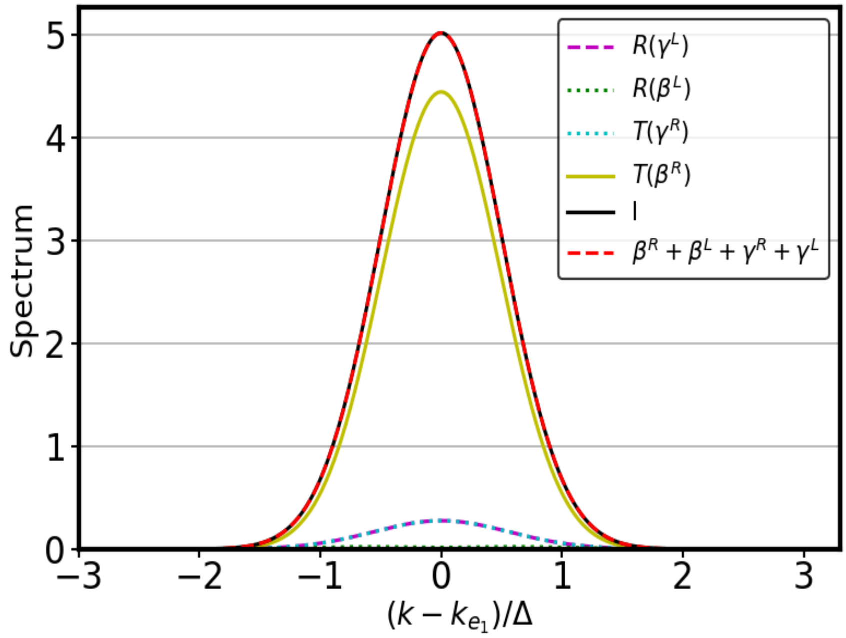

Figure 5.3: Spectra of the incident, reflected β and γ photons, and transmitted β and γ photons. The Rabi frequency $\Omega_R = 2$. The superscripts R and L represent the right and left propagating modes, respectively.

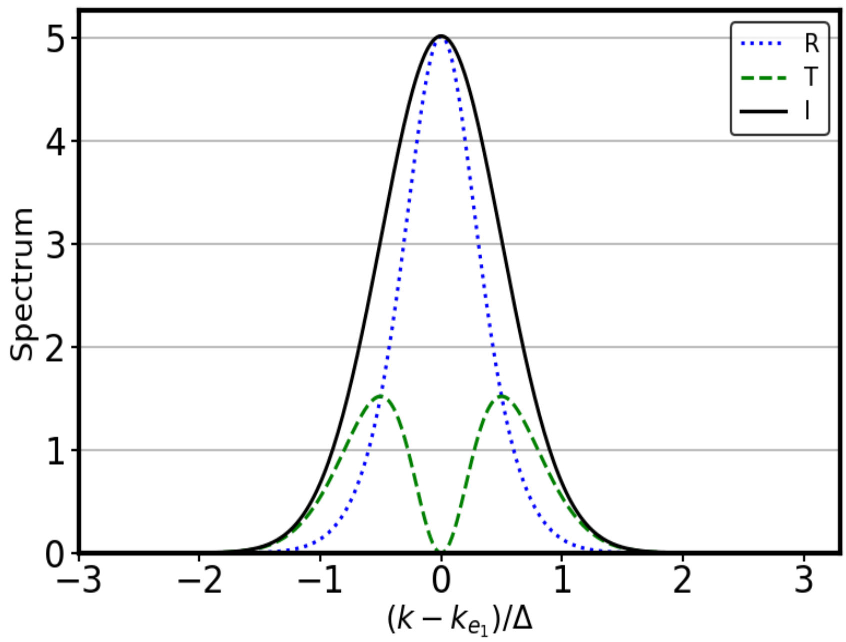

Figure 5.4: Spectra of the incident, reflected, and transmitted photons. The Rabi frequency $\Omega_R = 0$ which is the case of a two-level atom coupled to a 1-D waveguide.

The spectra of the reflected and transmitted β and photons are shown in Fig.5.3 when $\Omega_R = 2$. It is obvious that as the value of Rabi frequency is increased, the transmitted amount of the the incident pulse increases. We can conclude that the strength of the Rabi frequency plays a significant role in reflecting and transmitting the the incident photon pulse. When the Rabi frequency is off, the atom behaves as a two-level atom as shown in Fig. 5.4.

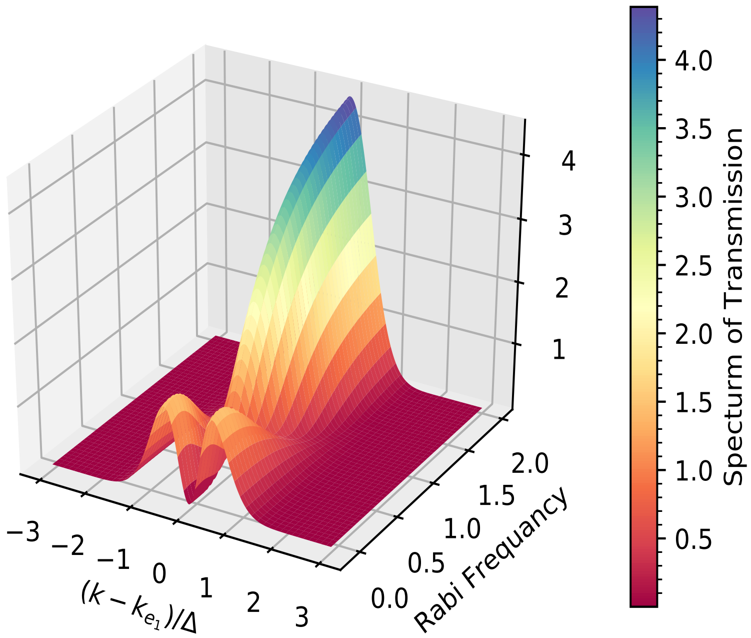

Figure 5.5: Transmission spectrum of the β photon at $t = 0$ for different Rabi Frequencies.

Fig. 5.5 shows how the transmission spectrum is affected by the Rabi frequency. When the Rabi frequency is zero, we see no photon is transmitted at resonance. Then, as the Rabi frequency is gradually increased, the transmission starts to increase.

5.3.2 Reflectivity and Transmittivity

Reflectivity and transmissivity of the β photon can be calculated using the following equations

$$R = \frac{L}{2\pi}\int_{-\infty}^{\infty} d\delta k \left|\beta_{\delta k}^{L}(t\to\infty)\right|^2, \tag{5.23a}$$

$$T = \frac{L}{2\pi}\int_{-\infty}^{\infty} d\delta k \left|\beta_{\delta k}^{R}(t\to\infty)\right|^2. \tag{5.23b}$$

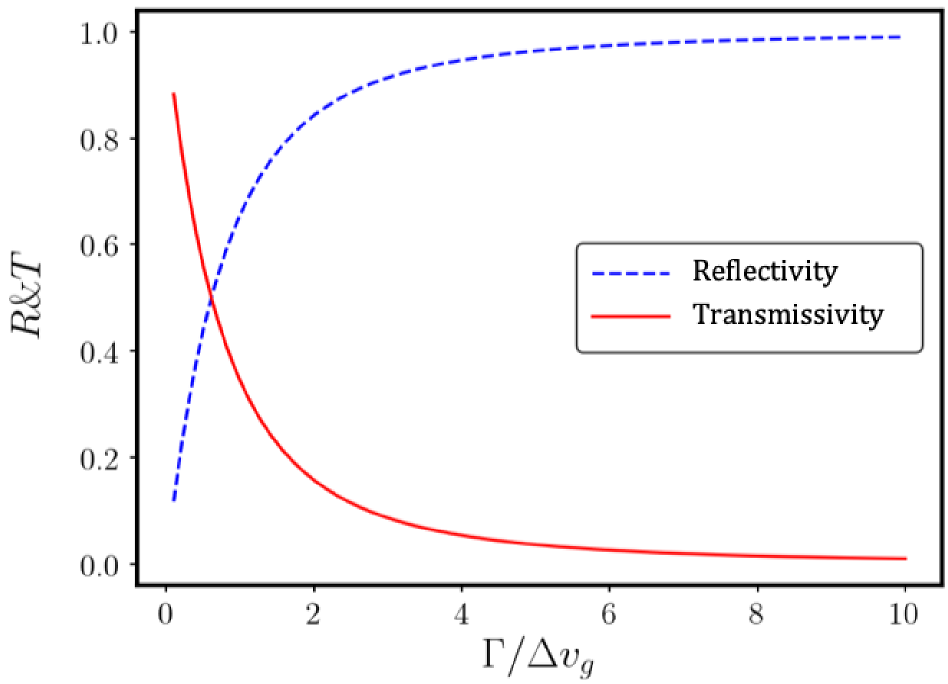

Figure 5.6: Reflectivity and transmissivity of the β when $\Omega_R = 0$. This situation is the same as that of a two-level atom coupled to a 1-D waveguide [4]

Fig.5.6 shows the reflectivity and transmissivity of the β when $\Omega_R = 0$, while the horizontal axis represents the coupling strength between the atom and the waveguide. This result agrees with the result of the reflectivity and transmissivity in the case of two-level atom.

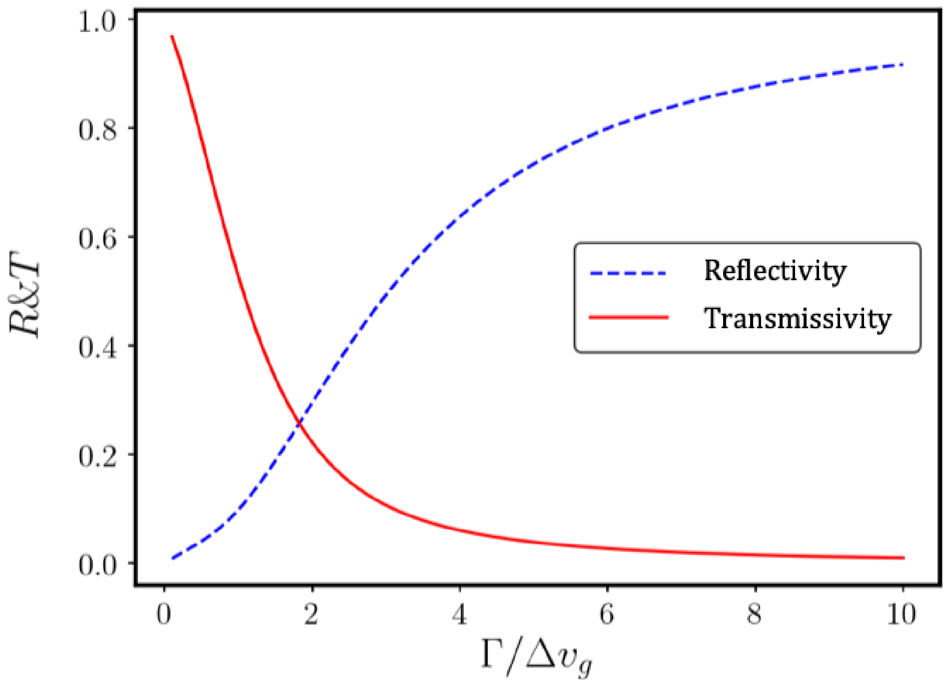

Figure 5.7: Reflectivity and transmissivity of the β when $\Omega_R = 1$.

Fig.5.7 shows reflectivity and transmissivity when the driving field is turned on and the Rabi frequency Omega is equal to 1. The reflectivity increases more slowly than the previous case where $\Omega_R = 0$.

In Fig.5.8 the Rabi frequency is increased still further. We see that the transmissivity decreases slowly while the reflectivity slowly increases.

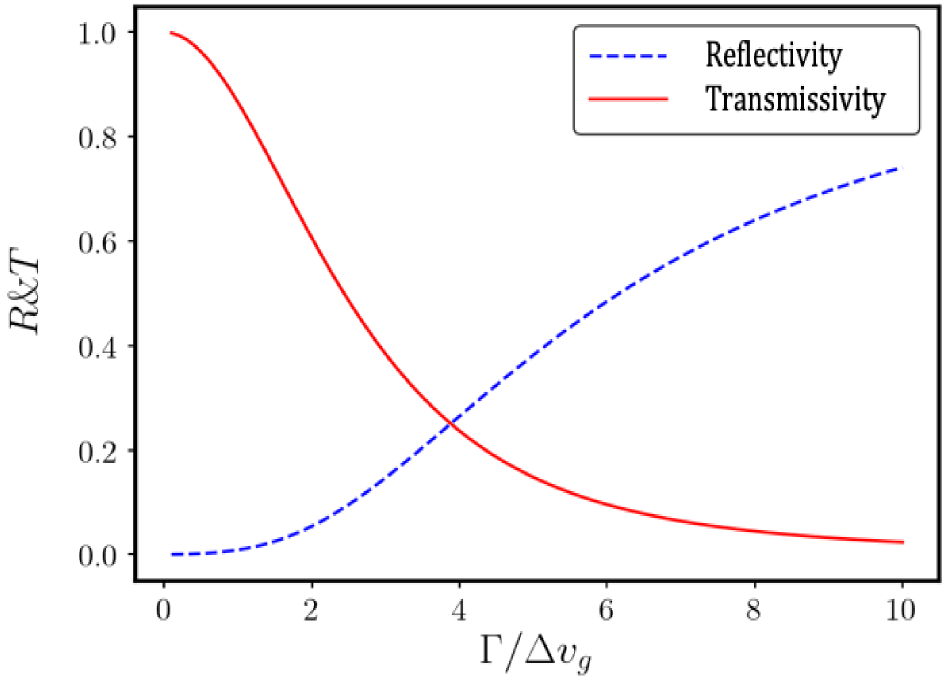

Figure 5.8: Reflectivity and transmissivity of the β when $\Omega_R = 2$.

5.3.3 Intensity

Another important quantity to consider is the photon pulse shape. The transmitted pulse shape can be calculated numerically using the equation

$$\beta_x^R(t) = e^{ik_{e_1}x} \int_{-\infty}^{\infty} d\delta k\ \beta_{\delta k}^R(t) e^{i\delta k(x - v_g t)}, \tag{5.24}$$

where $\beta_{\delta k}^R(t) = \beta(0) - i\sqrt{\frac{\Gamma v_g}{2L}} e^{ikr} \int_0^t dt' \alpha^{e_1}(t') e^{-i(\omega_{e_1} - \omega_k)t'}$, and the reflected pulse shape can be calculated numerically

$$\beta_x^L(t) = e^{-ik_{e_1}x} \int_{-\infty}^{\infty} d\delta k\ \beta_{\delta k}^L(t) e^{-i\delta k(x + v_g t)}, \tag{5.25}$$

where $\beta_{\delta k}^L(t) = -i\sqrt{\frac{\Gamma v_g}{2L}} e^{ikr} \int_0^t dt' \alpha^{e_1}(t') e^{-i(\omega_{e_1} - \omega_k)t'}$.

Fig. 5.9 shows the pulse shape of the the incident, reflected, and transmitted β photon

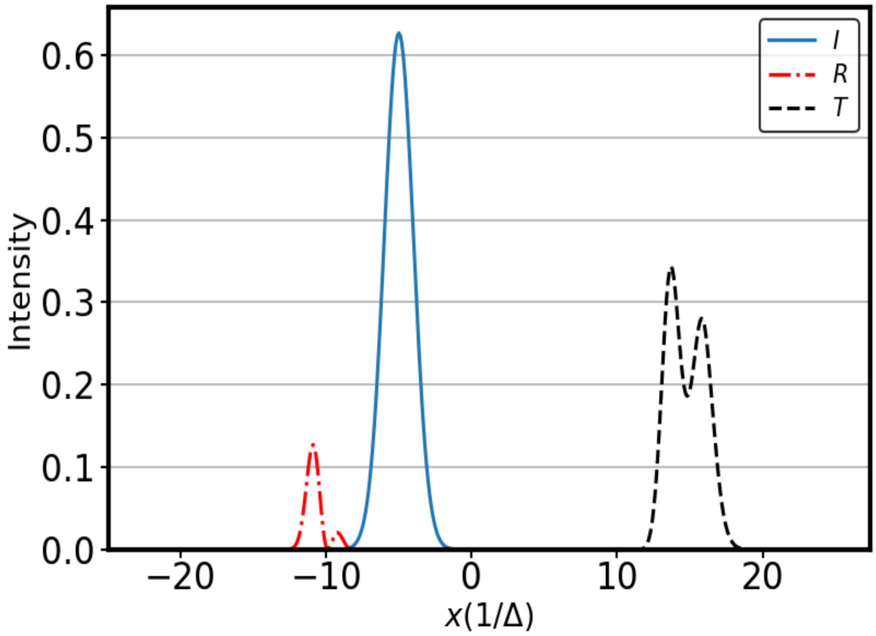

Figure 5.9: Pulse shape of the incident at $t = \frac{6}{\Delta}$, reflected at $t = \frac{15}{\Delta}$, and transmitted β photon $t = \frac{15}{\Delta}$. The Rabi frequency $\Omega_R = 1$.

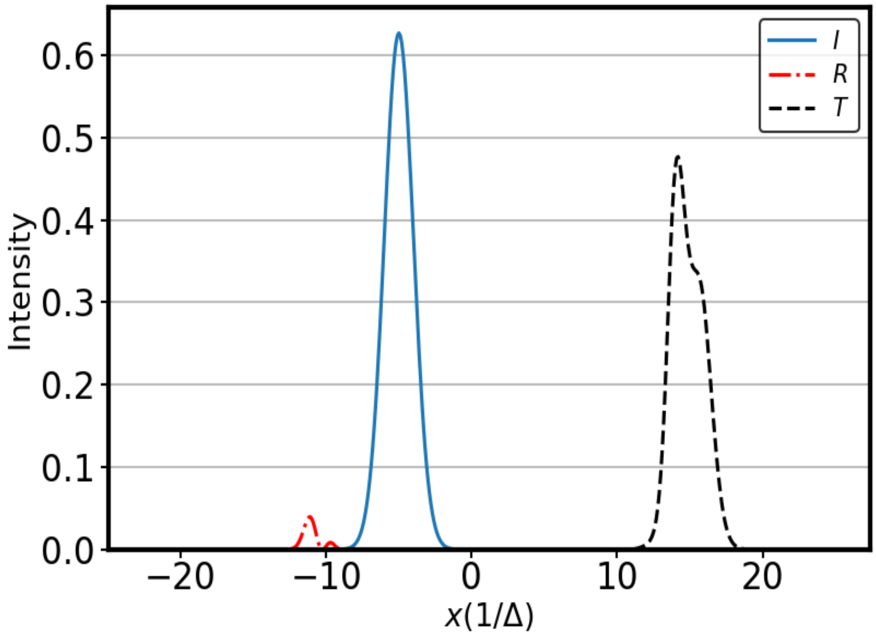

Figure 5.10: Pulse shape of the incident, reflected, and transmitted β photon. The Rabi frequency $\Omega_R = 2$.

when the value of the Rabi frequency is $\Omega_R = 1$ and the atom is placed at the origin $x = 0$. Fig. 5.10 shows the pulse shape of the the incident, reflected, and transmitted β photon when the value of the Rabi frequency is $\Omega_R = 2$. Fig. 5.9 shows the pulse shape

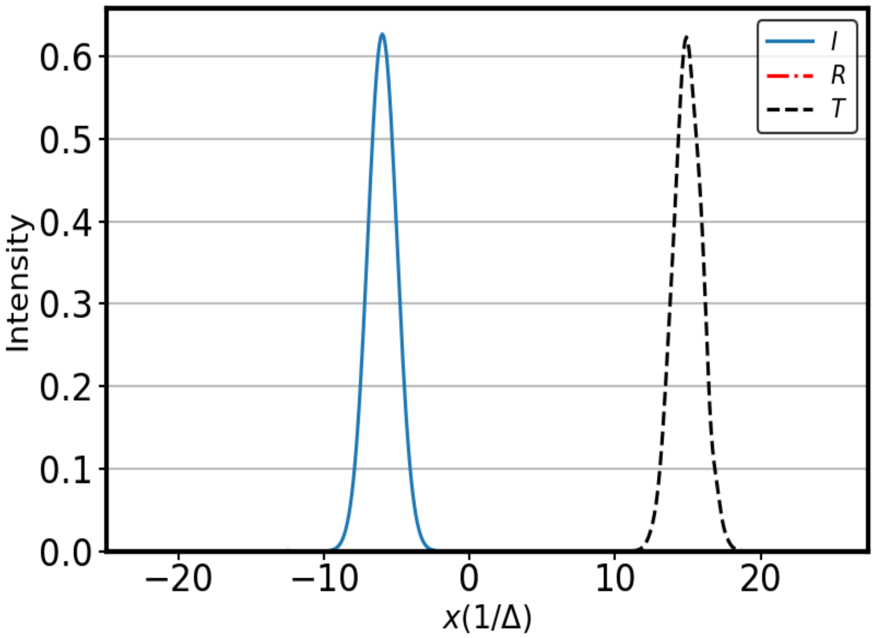

Figure 5.11: Pulse shape of the incident, reflected, and transmitted β photon. The Rabi frequency $\Omega_R = 5$.

of the the incident, and transmitted β photon when the value of the Rabi frequency is $\Omega_R = 5$. When the strength of external field is a strange the system acts as transparency material. Fig. 5.11

Chapter 6

Conclusion

In conclusion, we considered the propagation of a single-photon pulse in a 1-D waveguide coupled to a V-type three-level atom. We obtained the time-dependent solutions of the problem using the time-dependent Schrödinger equation. We showed that by varying only the frequency of the driving field, the reflection and transmission of the incoming photon pulse can be controlled. Therefore, the external driving field acts as an optical switch to control the propagation of the photon pulse in the waveguide. When the external driving field is switched off, a complete reflection of the incident photon pulse occurs. At some values of the Rabi frequency of the driving field, the system acts as a transparent medium and the photon pulse is transmitted through the waveguide.

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