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Explicit formulas for photon number discrimination with on/off detectors

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Discriminating between Fock states with a high degree of accuracy is a desirable feature for modern applications of optical quantum information processing. A well-known alternative to sophisticated photon number discriminating detectors is to split the field among a number of simple on/off detectors and infer the desired quantity from the measurement results. In this work we find an explicit analytical expression of the detection probability for any number of input photons, any number of on/off detectors, and we include quantum efficiency and a false count probability. This allows us to explicitly invert the conditional probability using Bayes' theorem and express the number of photons that we had at the input in the most unbiased way possible with ready-to-use formulas. We conclude with some examples.

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1. INTRODUCTION

For practical applications of optical quantum information processing, it would be a great advantage to have a detector that can discriminate between different photon number states [1,2]. There are currently several different solutions that allow one to achieve this to some extent [3–7], but the resources that such detectors require (such as very low temperatures, particular materials, and/or optical configurations) may make them costly to obtain and not straightforward to operate. There are workarounds that involve squeezing more information out of the conventional detectors [8,9], or by demultiplexing the photons in time or space and directing them toward one or multiple single-photon detectors [10–27].

The most common single-photon detectors are only able to tell us whether they detected "zero photons" or "more than zero photons." Furthermore, they are subject to noise and a sub-optimal efficiency, which means that sometimes they click when they should not have or that they do not click when they should have [28,29].

In this work we study photon-number discrimination by demultiplexing, and our novel contribution is explicit formulas that are straightforward to implement and that take into account quantum efficiency and noise, as well as any number of detectors.

2. DISCRIMINATION PROBABILITY

We consider a balanced linear device that converts D inputs into D outputs. A single-mode input then becomes

$$\hat{a}^\dagger_{\text{in}} \rightarrow \sum_{j=1}^{D} \frac{\hat{b}^\dagger_j}{\sqrt{D}}. \tag{1}$$

A possible physical model for this device can be a cascaded sequence of $D-1$ conventional beam splitters, with reflectivities $\frac{1}{D}, \frac{1}{D-1} \dots \frac{1}{2}$ [30], but other possibilities exist; for example, using a top-hat pulse with uniform spatial intensity [31], followed by an array of detectors. We note that all-optical solutions are just one area of applicability of our results, which can be applied to any demultiplexer with a final set of detectors, which can be even as large as the set of pixels in an electron multiplying charge-coupled device (EMCCD) or an intensified charge-coupled device (ICCD).

Note that a demultiplexer should avoid closed paths, because photons, being bosons, would undergo the Hong–Ou–Mandel effect and bunch together instead of spreading out into the available modes, which defeats the purpose of the demultiplexer. Furthermore, in the absence of closed paths we are not required to take phases into account and the problem reduces to a classical counting problem.

The demultiplexer finally couples to a set of on/off single photon detectors. We wish to calculate the probability of observing C clicks, given an initial photon number state of N photons and given that all D detectors have a quantum efficiency η and a dark count probability ε. We start from the ideal case $\eta = 1, \varepsilon = 0$ and then move on to the general case $0 \le \eta \le 1, 0 \le \varepsilon \le 1$ and from the general case we retrieve a simple corollary that holds for $0 \le \eta \le 1, \varepsilon \ll N/D$.

A. Ideal Detectors

The fundamental ingredient for our analysis is the probability of distributing N photons into exactly C out of D detectors. We start by numbering the detectors from 1 to D, then a certain string of numbers will describe an event, where the detectors numbered in the string are the ones that clicked. Note that in the absence of noise the number of clicks cannot exceed the number of input photons, i.e., $C \le N$.

Lemma (ideal detection). *The probability of observing C clicks by distributing a Fock state of N photons evenly among D ideal on/off detectors (i.e., noiseless and with 100% quantum efficiency) is given by*

$$P_D(C|N) = \binom{D}{C} \frac{C!}{D^N} \mathcal{S}_N^C, \tag{2}$$

where $\mathcal{S}$ is the Stirling number of the second kind.

Proof. Our goal is to compute the fraction of detection strings (i.e., the strings of numbers describing a detection event, as described above) that include exactly C out of D detectors, modulo reorderings.

Call S_i the set of strings corresponding to N input photons that do not include the ith detector. Then select a specific subset $\mathcal{K}$ of cardinality $|\mathcal{K}| = k$ from the D detectors. The set of strings that exclude the detectors in $\mathcal{K}$ is the intersection of the sets excluding each of the elements of $\mathcal{K}$: $\bigcap_{i \in \mathcal{K}} S_i$ and its cardinality is

$$\left| \bigcap_{i \in \mathcal{K}} S_i \right| = (D - k)^N, \tag{3}$$

as we have N choices with repetition, from $(D - k)$ possible detectors. Of course, we are also counting strings that exclude *any other* detector, in addition to the ones in $\mathcal{K}$. To get around this problem, we use the inclusion–exclusion rule to count the elements in unions of sets S_i. In particular, we need the union of S_i for $i \in \{1, \ldots, D\}$, i.e., the set of all strings that exclude *at least* one detector, whose cardinality is

$$\left| \bigcup_{i=1}^{D} S_i \right| = \sum_{j=1}^{D} (-1)^{j+1} \binom{D}{j} (D - j)^N. \tag{4}$$

The complement of this set is the set of strings that include all D detectors (if they missed any they would belong to $\bigcup_{i=1}^{D} S_i$), whose cardinality is

$$\left| \overline{\bigcup_{i=1}^{D} S_i} \right| = D^N - \left| \bigcup_{i=1}^{D} S_i \right| = \sum_{j=0}^{D} (-1)^j \binom{D}{j} (D - j)^N. \tag{5}$$

Finally, we can compute the number of strings that include precisely C out of D detectors: pick $D - C$ detectors to be excluded (there are $\binom{D}{C}$ ways of doing this) and compute the number of strings that include all of the remaining C detectors:

$$\binom{D}{C} \left| \overline{\bigcup_{i=1}^{C} S_i} \right| = \binom{D}{C} \sum_{j=0}^{C} (-1)^j \binom{C}{j} (C - j)^N \tag{6}$$

$$= \binom{D}{C} C! \mathcal{S}_N^C, \tag{7}$$

where $\mathcal{S}_N^C$ is the Stirling number of the second kind. So the probability of ending up with exactly C clicks is the result above divided by the total number of possible strings D^N:

$$P_D(C|N) = \binom{D}{C} \frac{C!}{D^N} \mathcal{S}_N^C, \tag{8}$$

and our proof is complete (see Eq. (14) in [27] for an implicit positive operator-valued measure (POVM) representation). □

B. Nonideal Detectors

Nonideal detectors are subject to mainly two effects: sub-unity quantum efficiency and noise, which can come from various sources. We model these as Bernoulli trials, where for each detector we have a probability η of missing the photon and a probability ε of a false count within the measurement window, in which case we are informed that the detector clicked regardless of a photon hitting it or not. Whether a detector detects an actual photon or gives a false count, we consider it unable to give further clicks until the electronics have enough time to reset (e.g., about 40 ns for avalanche photodiodes). In this section we take both of these effects into account.

Theorem (noisy detection). *The probability of observing C clicks by distributing a Fock state of N photons evenly among D on/off detectors with quantum efficiency η and false count probability ε, indicated as $P_{D,\eta,\varepsilon}(C|N)$, is given by*

$$\sum_{i=0}^{C} p_\varepsilon(i|D) \sum_{j=C-i}^{N} p_{\frac{D-i}{D}}(j|N) \sum_{k=C-i}^{j} p_\eta(k|j) P_{D-i}(C - i|k), \tag{9}$$

where $p_\xi(m|n) = \binom{n}{m} \xi^m (1 - \xi)^{n-m}$ is the probability of having m successes out of n trials when the success probability of a single trial is ξ.

Proof. The proof comprises of three steps, each of which is of a similar nature: we consider all the ways in which an event can happen and we sum the relative probabilities. In the first step we split the observed number of clicks into spurious and real clicks. In the second step we split the initial photons into those that landed onto inactive detectors (the noisy ones) and those that landed onto active ones. In the third step we split the photons that landed onto active detectors into those that were lost because of quantum efficiency and those that were not. Finally, we use the ideal detection Lemma.

Step 1: We sum over the probability of obtaining C total clicks by having i of them come from noise and $C - i$ come from actual detections. We write the probability of i false events given D detectors as $p_\varepsilon(i|D) = \binom{D}{i} \varepsilon^i (1 - \varepsilon)^{D-i}$.

Step 2: Now $C - i$ clicks must come from real detection events at the remaining $D - i$ active detectors. The probability that j out of N photons make it to the $D - i$ active detectors is $p_{\frac{D-i}{D}}(j|N)$.

Step 3: As our detectors have a quantum efficiency $\eta \le 1$, the probability of remaining with k out of j photons is given by $p_\eta(k|j)$.

Now we can now apply the Lemma to write the probability of detecting $C - i$ out of k survivor photons with $D - i$ detectors and combine these steps in the final result. □

There is a simple corollary of this theorem, which describes the case $\varepsilon = 0$. Such corollary can be used even for noisy detectors as long as the number of false positives is low enough ($D\varepsilon \ll N$):

Corollary (noiseless detection). *The probability of observing C clicks by distributing a Fock state of N photons evenly among D noiseless on/off detectors with quantum efficiency η is given by*

$$P_{D,\eta}(C|N) = \sum_{k=C}^{N} p_\eta(k|N) P_D(C|k). \quad \textbf{(10)}$$

Proof. We use the identity $p_0(m|n) = \delta_{m,0}$ to replace every occurrence of i in the noisy detection theorem by 0, and the identity $p_1(m|n) = \delta_{m,n}$ to replace every occurrence of j by N. This gets rid of the first two summations and the result follows. □

Note that modeling the imperfect detectors by placing a beam splitter with transmissivity η in front of ideal detectors [32,33] would be wrong in this context because quantum efficiency does not apply to false counts: first we exclude false counts and photons that landed on inactive detectors, only then we can factor in the quantum efficiency.

3. RETRODICTING THE PHOTON NUMBER

To retrodict the photon number *given* an observed number of clicks, we have to invert the probability in the main theorem using Bayes' rule:

$$P_{D,\eta,\varepsilon}(N|C) = \frac{P_{D,\eta,\varepsilon}(C|N)\,\Pr(N)}{\sum_k P_{D,\eta,\varepsilon}(C|k)\,\Pr(k)}. \quad \textbf{(11)}$$

This general formula is always valid, but it cannot be solved explicitly unless we specify the prior, which is what we will do next, for some special cases of particular relevance.

A. Poisson Prior

In the case of a Poissonian prior with mean photon number μ (which may occur when we deal with coherent states, for instance), we have

$$\Pr(N) = \frac{\mu^N e^{-\mu}}{N!}, \quad \textbf{(12)}$$

and we can find an explicit expression for the ideal retrodiction probability:

$$P_D^{\text{Poisson}}(N|C) = \frac{C!\mathcal{S}_N^C}{N!\gamma^N} \frac{1}{(e^{1/\gamma} - 1)^C}, \quad \textbf{(13)}$$

where $\gamma = D/\mu$.

B. Thermal Prior

In the case of a thermal prior with mean photon number μ (which occurs, for instance, for two-mode squeezed vacuum states), we have

$$\Pr(N) = \frac{\mu^N}{(\mu+1)^{N+1}}, \quad \textbf{(14)}$$

and the ideal retrodiction probability can be written as

$$P_D^{\text{Therm}}(N|C) = \frac{C!\mathcal{S}_N^C}{(D+\gamma)^N} \frac{\Gamma(D+\gamma)}{\Gamma(D+\gamma-C)!}. \quad \textbf{(15)}$$

C. Considerations

When one moves away from the ideal case, quantum efficiency typically matters more than the number of detectors. The probability of detecting all the input photons with a noiseless apparatus saturates at a value lower than 1 even for an infinite number of detectors:

$$\lim_{D\to\infty} P_{D,\eta}(N|N) = \eta^N. \quad \textbf{(16)}$$

The effect of noise in the detectors is tangible only when their number is sufficiently large, for instance, when the number of spurious counts is comparable with the actual number of photons hitting the detectors, i.e., when $D\varepsilon \approx N$. This fact makes the noiseless detection corollary a good tool even in the case of realistic detectors if we have a large enough number of them.

4. APPLICATIONS

We now would like to give a few examples of how to apply our results. The examples will be about retrodicting the photon number in order to herald some desired quantum states and are based on our analytical results (not on Monte Carlo simulations).

A. Example 1: Heralding of a NOON State

NOON states are two-mode states in the form $(|N,0\rangle + |0,N\rangle)/\sqrt{2}$. For this example we consider the following setup: we replace the two mirrors in the middle of a Mach–Zehnder (MZ) interferometer with 50:50 beam splitters and add detectors to measure the photons that leak. This configuration (if the phase difference between the two arms of the MZ is set to $\pi/2$) will output a $(|4,0\rangle + |0,4\rangle)/\sqrt{2}$ state if we start with the state $|3,3\rangle$ and if each of the two detectors measures exactly one photon.

Now the question is how well do we know that we had exactly one photon at the detectors? If we resort to demultiplexed detection, we first need to compute the prior joint probability $\Pr(N_1, N_2)$ of having N_1 photons at detector 1 and N_2 photons at detector 2. This is achieved using simple input–output relations for 50:50 beam splitters; we report it in Fig. 1.

Then, we apply Bayes' rule (assuming that the two sets of demultiplexed detectors are identical, but we could easily modify the equation below to account for different configurations) and find $P_{D,\eta,\varepsilon}(N_1, N_2|C_1, C_2)$ to be given by

$$\frac{P_{D,\eta,\varepsilon}(C_1|N_1)P_{D,\eta,\varepsilon}(C_2|N_2)\,\Pr(N_1,N_2)}{\sum_{k_1,k_2} P_{D,\eta,\varepsilon}(C_1|k_1)P_{D,\eta,\varepsilon}(C_2|k_2)\,\Pr(k_1,k_2)}. \quad \textbf{(17)}$$

We finally use the quantity $P_{D,\eta,\varepsilon}(N_1, N_2|C_1, C_2)$ to infer the retrodictive power of our demultiplexed detectors. To complete the example, in Fig. 2 we plot the retrodicted probabilities of four configurations: 4 and 64 detectors with 60% and 75% quantum efficiency (and 500 dark counts/s, with 10 ns gated measurement window), given that they both reported a single click each.

For comparison, in Fig. 3 we plot the retrodiction probabilities for a nondemultiplexed measurement.

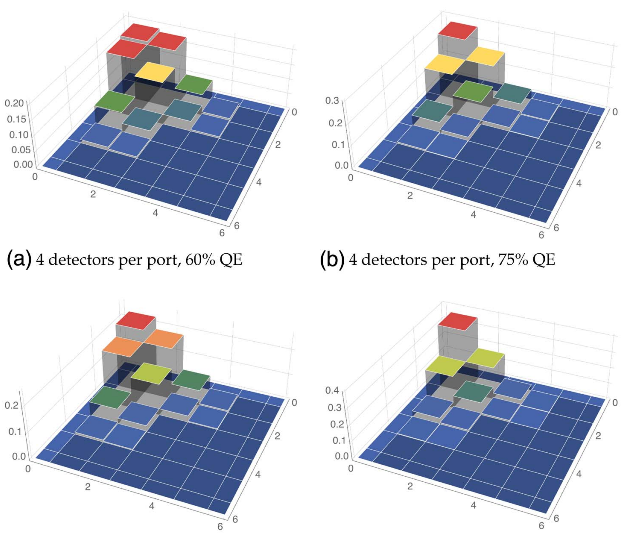
	0	1	2	3	4	5	6
0	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{3}{32}$	$\frac{7}{64}$	$\frac{39}{512}$	$\frac{15}{512}$	$\frac{5}{1024}$
1	$\frac{3}{64}$	$\frac{3}{64}$	$\frac{3}{64}$	$\frac{3}{128}$	$\frac{3}{512}$	0	0
2	$\frac{3}{32}$	$\frac{3}{64}$	$\frac{9}{256}$	$\frac{3}{256}$	$\frac{3}{1024}$	0	0
3	$\frac{7}{64}$	$\frac{3}{128}$	$\frac{3}{256}$	0	0	0	0
4	$\frac{39}{512}$	$\frac{3}{512}$	$\frac{3}{1024}$	0	0	0	0
5	$\frac{15}{512}$	0	0	0	0	0	0
6	$\frac{5}{1024}$	0	0	0	0	0	0

Fig. 1. Joint probabilities of having (i, j) photons (where i and j are listed in the headings on top and on the left) at the detectors in the modified MZ interferometer of the NOON state heralding example. These are computed assuming the input $|3, 3\rangle$.

B. Example 2: Single-Photon Heralding from Squeezed Vacuum

We now consider an example of single-photon heralding from a two-mode squeezed vacuum, which is performed by producing photons in pairs and heralding one by detecting the other. Such a two-mode state can be generated by pumping a nonlinear crystal with an intense coherent laser [34]. The output of the process is a state in the following form:

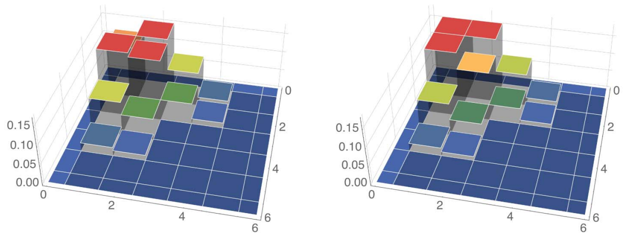
$$\hat{S}(\zeta)|0, 0\rangle = \sum_{n=0}^{\infty} e^{in\phi} \frac{\sinh(g)^n}{(\sinh(g)^2 + 1)^{\frac{n+1}{2}}} |n, n\rangle, \quad (18)$$



(a) 4 detectors per port, 60% QE (b) 4 detectors per port, 75% QE

(c) 64 detectors per port, 60% QE (d) 64 detectors per port, 75% QE

Fig. 2. Plots of the probability of retrodicted photon number for a NOON state heralding setup using demultiplexed detection. Although the most probable case is the desired $|1, 1\rangle$, its individual probability can be quite low, which leads to a low fidelity with the desired NOON state. The bottleneck in this case is quantum efficiency: even increasing the number of detectors from 4 to 64 does not perform as well as increasing the quantum efficiency from 60% to 75%.



(a) 1 detector per port, 75% QE (b) 1 detector per port, 100% QE

Fig. 3. (Left) Pair of realistic detectors are likely to lie: if they report a single click each, the state was more likely to be $|1, 2\rangle$ or $|2, 1\rangle$ or even $|2, 2\rangle$. (Right) Even a pair of ideal (100% quantum efficiency) detectors assign equal probability to the states $|1, 1\rangle$, $|1, 2\rangle$, and $|2, 1\rangle$.

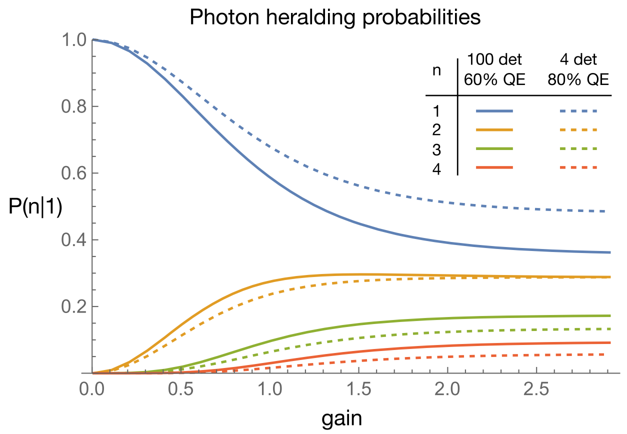


Fig. 4. Plots of the probability of retrodicted photon number for a squeezed vacuum state. Again, the bottleneck is quantum efficiency: four detectors with 80% quantum efficiency are better at identifying a single photon than 100 detectors with 60% quantum efficiency.

where $\zeta = ge^{i\phi}$ is the squeezing parameter. For small enough values of the gain g one can indeed ignore components with photon number larger than one, but if the gain is too large, the heralded state is likely to contain more than one photon. If such states were further used for crucial applications, such as quantum cryptography, they would be vulnerable, for example, to the photon number splitting attack. Could a demultiplexed detection scheme make for a better heralded single-photon source? First note that the amplitudes of the two-mode squeezed vacuum follow a thermal distribution, if we recognize that $\sinh(g)^2$ is the mean photon number per mode. Then, we apply Eq. (11) [we could use Eq. (15) in case our quantum efficiency is high] to find the retrodicted photon number distribution, which we plot for a few examples in Fig. 4. Note that as the gain increases, the probability of the various number states levels off and becomes constant; but recall that these probabilities are conditional on detecting a single photon, whose probability will decrease with gain.

5. CONCLUSIONS AND OUTLOOK

In conclusion, we have shown the most unbiased way of analyzing a detection event in a demultiplexed measurement scheme, taking noise and efficiency into account. The corollary of our theorem can apply even to realistic situations if some conditions on the noise are met, which can be very advantageous as it is computationally much simpler to implement than the full theorem.

Our results can be applied also to optical engineering issues, such as on-chip denoising in consumer imaging devices, where multiple pixels can fill an Airy disk and can be used to retrodict the intensity more accurately. There are still interesting questions to be asked, for instance, whether it is possible to find closed form solutions of Eq. (11) for useful priors when the quantum efficiency is not unity, or if there is a reasonable way of relaxing the assumption of uniform illumination. We leave these to a future work.

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