Recovering full coherence in a qubit by measuring half of its environment

Filippo M. Miatto,[1,2] Kevin Piché,[2] Thomas Brougham,[3] and Robert W. Boyd[2,3,4]
[1]*Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*
[2]*Department of Physics, University of Ottawa, Ottawa, Ontario, Canada K1N 6N5*
[3]*School of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, United Kingdom*
[4]*Institute of Optics, University of Rochester, Rochester, New York 14627, USA*
(Received 20 March 2015; published 18 December 2015)

When a quantum system interacts with its environment it may incur in decoherence. Quantum erasure makes it possible to restore coherence in a system by gaining information about its environment, but measuring the whole of it may be prohibitive: Realistically, one might be forced to address only an accessible subspace and neglect the rest. In such a case, under what conditions will quantum erasure still be effective? In this work we compute analytically the largest recoverable coherence of a random qubit plus environment state and we show that it approaches 100% with overwhelmingly high probability as long as the dimension of the accessible subspace of the environment is larger than $\sqrt{D}$, where D is the dimension of the whole environment. Additionally, we find a sharp transition between a linear behavior and a power-law behavior as soon as the dimension of the inaccessible environment exceeds the dimension of the accessible one. Our results imply that the typical states of a qubit plus environment system admit a measurement spanning only about $\sqrt{D}$ degrees of freedom, any outcome of which projects the qubit on a maximally coherent state. This suggests, for instance, that in the dynamics of open quantum systems, if the interactions are known, it would in principle be possible to gain sufficient information and restore coherence in a qubit by dealing with a fraction of the physical resources.

DOI: 10.1103/PhysRevA.92.062331 PACS number(s): 03.67.−a, 03.65.Yz

I. INTRODUCTION

Decoherence is a physical process that interests the scientific community from a fundamental point of view (how does the quantum-to-classical transition occur?) and also from a technical one (how can we preserve the coherence of a system?) [1–5]. One of the techniques for restoring coherence in a quantum system is known as quantum erasure, which consists in measuring the environment of the system in the most appropriate basis in order to erase the information that it stores and thereby recover coherence [6]. The typical pedagogic example consists in a Young double-slit experiment where two orthogonal polarizers have been put in front of the slits and the interference fringes have disappeared. Quantum erasure would consist of orienting a polarizing beam splitter diagonally before the screen to erase the which-slit information stored in the polarization (which was acting as the environment) and sort the ensemble of photons into two subsets, each of which displaying full fringes. Coherence is an important notion in quantum physics, which one can find in seemingly disparate areas such as thermodynamics [7,8], reference frames, and conservation laws [9,10], as well as operational frameworks [11].

Despite the sophistication of current experimental techniques, at a certain point in the complexity of the environment and of its interactions with our systems, it becomes no longer possible to handle all the relevant degrees of freedom and coherence degrades irreversibly. This technical limitation motivates our work, where we consider a single qubit $\mathcal{Q}$, immersed in an environment with Hilbert space $\mathcal{A}\otimes\mathcal{K}$, where the A-dimensional subspace $\mathcal{A}$ is accessible and the K-dimensional subspace $\mathcal{K}$ is inaccessible (see Fig. 1).

Although notions of coherence that apply to larger systems exist [12], there are several situations in which a single qubit takes the leading role, e.g., each of the individual quantum signals in most models of quantum key distribution [13], an individual spin in a nitrogen-vacancy center in diamond [14–16], an individual atom in an optical lattice [17,18], and a single ion in a magnetic trap [19], as well as qubits made of collective quantum systems such as flux and superconducting qubits [20,21]. For this reason, in the present work we concentrate on the coherence of a single qubit; we postpone the study of the coherence retrieval of larger systems to a future study.

The rest of this paper is organized as follows. In the next section we define the coherence of a qubit and we show how it is influenced by a general measurement on its environment. Then we split the environment into an accessible and an inaccessible part, i.e., we restrict our measurement operators to be in the form $\pi_j = \pi_j^{\mathcal{A}} \otimes \mathbb{1}^{\mathcal{K}}$, and we prove the main result. In the final section we supply some examples in terms of qubit ensembles and we show that the average coherence becomes a typical quantity as the environment grows in size.

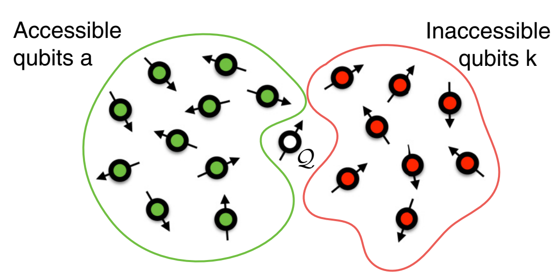

FIG. 1. (Color online) We imagine a qubit $\mathcal{Q}$ within an ensemble of n environment qubits, where a of them are accessible. The rest $k = n - a$ are inaccessible. We find that if $a \geqslant k$ (i.e., if we can access at least half of them) there exists an optimal measurement on the accessible qubits whose outcomes project $\mathcal{Q}$ onto states with coherence $\langle\mathcal{C}\rangle \sim 1 - \frac{1}{4}2^{k-a}$, which approaches 1 exponentially fast as $a \to n$.

II. QUANTIFYING AND RESTORING COHERENCE

If one deals with systems with a Hilbert space of dimension greater than 2, it is possible to choose from different definitions of coherence (for a thorough review see Ref. [12]). However, for a qubit they are all equivalent, so we will avoid mentioning them all and concentrate on a single one. The coherence of a qubit $\mathcal{Q}$ in the Bloch sphere picture is the distance of the Bloch vector from the imaginary line connecting the north pole to the south pole, i.e., given a Bloch vector with coordinates $\mathbf{v} = (x,y,z)$, the coherence is $\sqrt{x^2+y^2}$, also known as visibility in view of the analogy of a qubit with the state of a photon in a Mach-Zehnder interferometer [22]. It is clear that coherence is basis dependent: If we picked a different pair of opposite points on the surface as the new north and south poles, the coherence would generally change. In terms of the density matrix of the qubit, the coherence is given by twice the absolute value of either of the off-diagonal elements.

In general, quantum erasure relies on an optimal measurement of the environment $\mathcal{E}$ of a system $\mathcal{Q}$ to restore its coherence. Such a measurement consists of a certain number of measurement operators $\{\pi_j^{\mathcal{E}}\}$, where the jth measurement operator $\pi_j^{\mathcal{E}}$ applied to the environment leaves $\mathcal{Q}$ in a state ρ_j with some probability p_j. Each state ρ_j displays a certain coherence $\mathcal{C}_j$ (defined below) [23]. It is important to note that quantum erasure does not rely on postselection [24–26], as the coherence that we will maximize is the average over all the outcomes: $\sum_j p_j\mathcal{C}_j$. Moreover, notice that the incoherent sum $\sum_j p_j\rho_k$ is the same as tracing over the environment: This prevents us from measuring coherence directly and violating the no-signaling principle. It is clear that the more information one is able to erase from the environment, the more of the original coherence one is able to restore.

When a qubit $\mathcal{Q}$ is entangled with another system (which we call $\mathcal{E}$ and which is not necessarily another qubit) we have a state in the following form:

$$|\psi\rangle = \alpha|0,e_0\rangle + \beta|1,e_1\rangle, \tag{1}$$

where $|e_0\rangle$ and $|e_1\rangle$ are the states of $\mathcal{E}$ corresponding to the states of $\mathcal{Q}$. The density matrix of $\mathcal{Q}$ alone is obtained by tracing over $\mathcal{E}$:

$$\rho_{\mathcal{Q}} = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta\langle e_1|e_0\rangle \\ \alpha\beta^*\langle e_0|e_1\rangle & |\beta|^2 \end{pmatrix}. \tag{2}$$

The coherence is therefore given by $\mathcal{C} = 2|\alpha\beta^*\langle e_0|e_1\rangle|$, which is proportional to the overlap between the states of $\mathcal{E}$. This happens because the more the states $|e_0\rangle$ and $|e_1\rangle$ are orthogonal, the better one can distinguish them and learn about the qubit, i.e., the more information about the alternatives of $\mathcal{Q}$ is stored in $\mathcal{E}$, which is acting as an environment. This is the essence of the duality principle.

A way of erasing such information would be to measure $\mathcal{E}$ in a basis that is unbiased with respect to $|e_0\rangle$ and $|e_1\rangle$, by way of an optimal measurement with elements $\pi_j^{\mathcal{E}}$ [23] defined by maximizing the mean coherence over all the probability operator measures on $\mathcal{E}$:

$$\langle\mathcal{C}\rangle = \sup_{\{\pi_j\}\in\mathrm{POM}(\mathcal{E})} \sum_j |2\alpha\beta^*\langle e_0|\pi_j|e_1\rangle| \tag{3}$$

$$= 2\,\mathrm{Tr}|\alpha\beta^*|e_1\rangle\langle e_0|| = 2|\alpha\beta^*|, \tag{4}$$

where $\mathrm{Tr}|x|$ is the trace norm of x. In this way, regardless of the outcome, we would learn as little as possible about which of the states $|e_0\rangle$ or $|e_1\rangle$ the environment is in and after such measurement the qubit is in a state with coherence $\mathcal{C} = 2|\alpha\beta^*|$. However, in realistic situations such flexibility may not be possible, i.e., we may not be able to access all the necessary degrees of freedom of $\mathcal{E}$ and we will have to split it into an accessible part and an inaccessible one. How much coherence can we expect to recover in that case?

To begin our study, we consider a random pure state of a qubit in an AK-dimensional environment with Hilbert space $\mathcal{A}\otimes\mathcal{K}$ that is split into an accessible subspace of dimension A and an inaccessible one of dimension K. Such a pure state is therefore sampled uniformly in a Hilbert space $\mathcal{Q}\otimes\mathcal{A}\otimes\mathcal{K}$ of dimension $2AK$. After tracing away the inaccessible environment $\mathcal{K}$, we are left with a $2A$-dimensional state in $\mathcal{Q}\otimes\mathcal{A}$ that can always be written as a $2A\times 2A$ density matrix

$$\rho = \begin{pmatrix} R_0 & X \\ X^\dagger & R_1 \end{pmatrix}, \tag{5}$$

where $\mathrm{Tr}(R_0)$ and $\mathrm{Tr}(R_1)$ are the probabilities of measuring the qubit in the alternatives $|0\rangle$ and $|1\rangle$ and X is the cross term. The largest coherence of the qubit that we can obtain by optimally measuring the accessible space $\mathcal{A}$ is given by twice the trace norm of the cross term $\mathcal{C} = 2\,\mathrm{Tr}|X|$ [see Eq. (4) in Ref. [23]]. We recall that the trace norm of X can be computed as the sum of the square roots of the A eigenvalues of the matrix $X^\dagger X$, i.e.,

$$\mathrm{Tr}|X| = \sum_{i=1}^{A}\sqrt{\lambda_i(X^\dagger X)}. \tag{6}$$

The random $2A$-dimensional states ρ are statistically distributed according to the induced trace measure $P_{2A,K}(\rho)$ and constitute a Ginibre ensemble [27]. A way of sampling uniformly from such an ensemble is to generate a $2A\times K$ complex Gaussian random matrix μ (with entries sampled from the complex normal distribution centered on the origin and with unit variance) and then build the $2A\times 2A$ density matrix

$$\rho = \frac{\mu^\dagger\mu}{\mathrm{Tr}(\mu^\dagger\mu)}. \tag{7}$$

However, when we calculate $\mathcal{C}$ we do not need the whole matrix ρ, but only the $A\times A$ off-diagonal block, which is proportional to the product $M = \mu_1^\dagger\mu_2$ of two independent $A\times K$ complex Gaussian random matrices μ_1 and μ_2 (see Fig. 2). We find the proportionality factor by recalling that we are averaging over

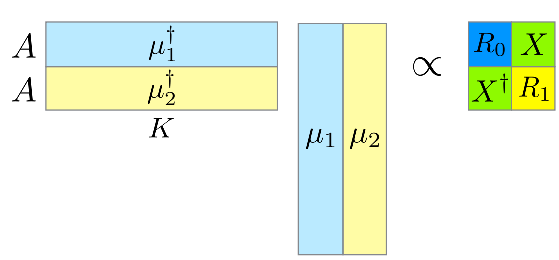

FIG. 2. (Color online) The $A\times A$ cross term X in the random matrix ρ of Eqs. (5) and (7) is proportional to the product between two independent random matrices μ_1 and μ_2 that make up μ.

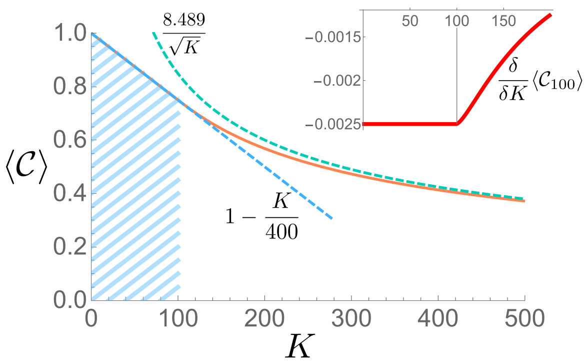

FIG. 3. (Color online) Average coherence $\langle \mathcal{C}\rangle$ for $A=100$ (solid orange line), together with the high-K and low-K approximations (dashed lines). Up to $K=A$, the behavior of $\langle \mathcal{C}\rangle$ is purely linear $1-\frac{K}{4A}$ (see the inset). For $K>A$ the behavior changes dramatically and is asymptotic to $O(1/\sqrt{K})$. The shaded indicates the linear region.

the whole ensemble and that the mean is linear, so we can take the average value of the denominator in (7): $\langle \mathrm{Tr}(\mu^\dagger \mu)\rangle = 4AK$. To find the average of $\mathrm{Tr}|M|$, we use the moments m_ℓ of the marginal distribution of eigenvalues of $M^\dagger M$. In particular, the average square root of the eigenvalues of $M^\dagger M$ is proportional to the moment of order $\ell = 1/2$, i.e., $\langle \mathrm{Tr}|M|\rangle = Am_{1/2}$. We can compute such a moment by applying Eq. (57) of Ref. [28] to our matrices and we find

$$m_{\frac{1}{2}} = \frac{4\pi^{5/2}(-1)^K\, {}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{2},1-A,1-A,1-K\\ \frac{1}{2}-A,\frac{1}{2}-A,\frac{1}{2}-K\end{matrix}\middle|1\right)}{A!\,\Gamma(A)\Gamma(K)}, \tag{8}$$

where the function ${}_4\tilde{F}_3$ is a regularized hypergeometric function. It is now straightforward to obtain the final result

$$\langle \mathcal{C}\rangle = 2\langle \mathrm{Tr}|X|\rangle = 2\frac{\langle \mathrm{Tr}|M|\rangle}{4AK} = \frac{m_{1/2}}{2K}. \tag{9}$$

III. EXAMPLES

An expression with regularized hypergeometric functions such as Eq. (8) can be rather obscure. For this reason, we evaluate it explicitly for some values of A and we show the two radically distinct behaviors that it exhibits, for $K\to\infty$ and for $1 \leqslant K \leqslant A$.

As the first example let us assume we have no control whatsoever over the environment. This is equivalent to setting $A=1$ in Eqs. (8) and (9) and we find

$$\langle \mathcal{C}_1\rangle = \frac{\pi^{3/2}(-1)^K}{2K!\,\Gamma\left(\frac{1}{2}-K\right)} \sim \frac{\sqrt{\pi}}{2\sqrt{K}} \quad \text{as } K\to\infty. \tag{10}$$

In other words, decoherence in the absence of any intervention scales like $O(1/\sqrt{K})$ at a rate of $\sqrt{\pi}/2$. As the second example we control a two- and three-dimensional space (i.e., $A=2$ and

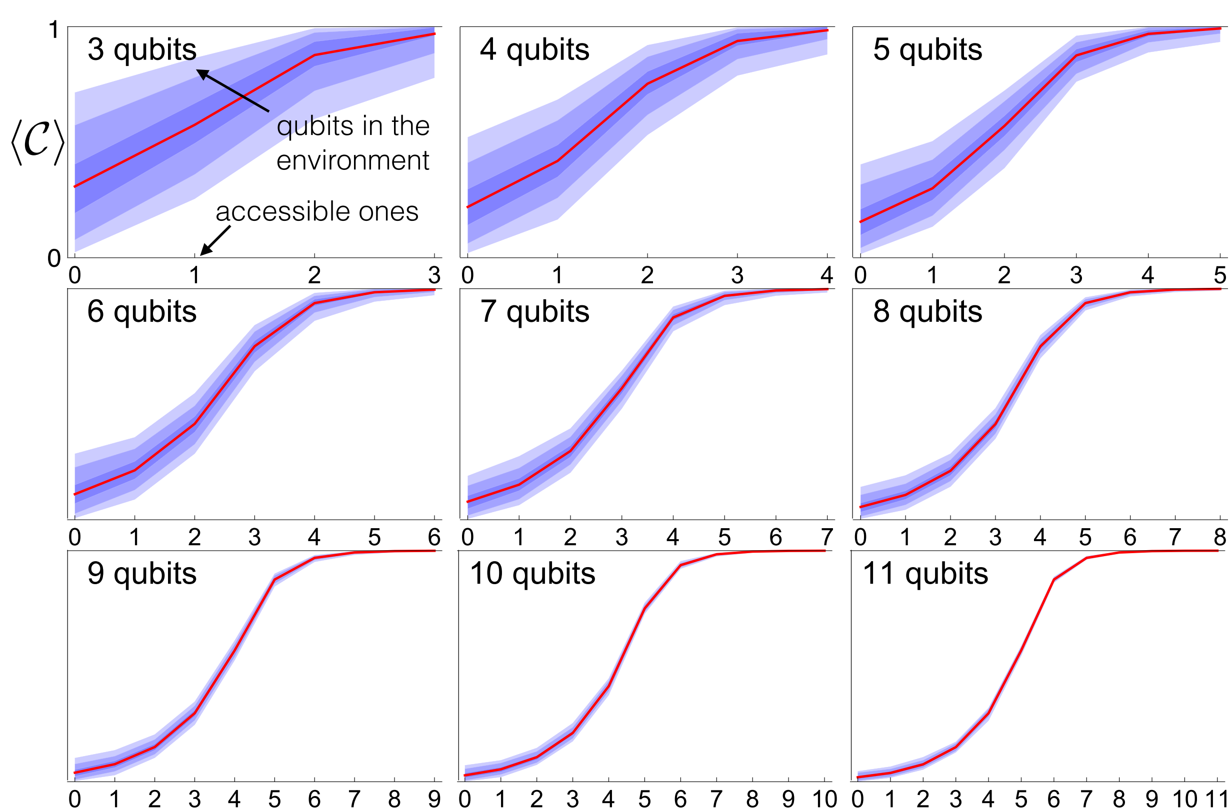

FIG. 4. (Color online) Average recoverable coherence $\langle \mathcal{C}\rangle$ of a qubit as a function of the number a of qubits of the environment that we can access, where the total n is displayed on top. In blue we show the 50th, 90th, and 99th percentiles around the mean (red line) and all plots are from 0 to 1. We see that $\langle \mathcal{C}\rangle$ transitions from a value close to 0 to a value close to 1 as we gain access to more than half of the environment qubits. Notice that as the total number of environment qubits grows, the mean becomes a better representative of the whole ensemble.

$A = 3$) and we find

$$\langle \mathcal{C}_2 \rangle = \frac{\pi^{3/2}(-1)^K(13-22K)}{32K!\Gamma\left(\frac{3}{2}-K\right)} \sim \frac{11\sqrt{\pi}}{16\sqrt{K}} \quad \text{as } K \to \infty, \tag{11}$$

$$\langle \mathcal{C}_3 \rangle \sim \frac{107\sqrt{\pi}}{128\sqrt{K}} \quad \text{as } K \to \infty, \tag{12}$$

and so on for larger accessible spaces. It turns out that the high-K scaling is always $O(1/\sqrt{K})$. If instead we look at the scaling for low values of K we find a linear behavior

$$\langle \mathcal{C}_A \rangle = 1 - \frac{K}{4A}, \quad 0 \leqslant K \leqslant A, \tag{13}$$

the transition happening rather sharply at $A = K$ (we give an explicit example for $A = 100$ in Fig. 3).

We now give a final example in terms of ensembles of qubits. Let us consider our qubit $\mathcal{Q}$ immersed in an ensemble of n other qubits and let us pick a random state of all $n+1$ of them. We wish to compute how much coherence we can expect to recover on $\mathcal{Q}$ on average as we gain control of more and more qubits in the ensemble. In this case $A = 2^a$ and $K = 2^{n-a} = 2^k$. We see that $\langle \mathcal{C} \rangle$ is close to zero for $a \lesssim k$ and it approaches 1 as $a \gtrsim k$ (see Figs. 4 and 5). Note that such variation always happens across the same number of qubits. In fact, for $a \geqslant \lceil n/2 \rceil$, the linear scaling makes it possible to compute the asymptotic behavior $\langle \mathcal{C} \rangle \sim 1 - \frac{2^k}{2^{a+2}}$ as $n \to \infty$. This law allows us to plot graphs for large numbers of qubits (see Fig. 5).

How typical is the value of $\langle \mathcal{C} \rangle$? To answer this question qualitatively we produced tens of thousands of random states of a qubit in an environment from $n = 3$ to $n = 11$ qubits. The results are shown in Fig. 4: The larger the environment, the more the average coherence becomes a typical property of the ensemble, as all the random states have a value of $\mathcal{C}$ that falls extremely close to $\langle \mathcal{C} \rangle$. Furthermore, we do not need extremely large environments: Already for a handful of qubits, the 99th percentile bands are very tightly squeezed around the mean.

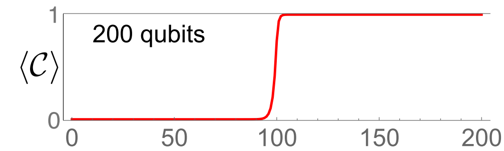

FIG. 5. (Color online) The recoverable coherence $\langle \mathcal{C} \rangle$ of a qubit $\mathcal{Q}$ for an environment of $n = 200$ qubits displays a very sharp increase from 0 to 1 as soon as one can control more than 100 of them, meaning that there exists an environment observable with elements $\{\pi_j^A \otimes \mathbb{1}^K\}$, each of which projects $\mathcal{Q}$ onto maximally coherent states.

IV. CONCLUSION

In this work we showed that quantum erasure can restore full coherence in a qubit even by addressing only about half of its environment. We also observed that the average recoverable coherence is a typical property of an ensemble of random states, i.e., $\mathcal{C} \sim \langle \mathcal{C} \rangle$ as $n \to \infty$. This means that the existence of such optimal measurement is guaranteed with overwhelming probability. Our result is even more surprising if restated in terms of degrees of freedom, as the optimal measurement need only address about $\sqrt{D}$ of the total number D of degrees of freedom of the environment. Moreover, typicality ensures that this result holds for any such partition of the environment; this phenomenon suggests a relation to quantum Darwinism [4].

ACKNOWLEDGMENTS

F.M.M. thanks Carlos González Guillén for a fruitful consultation, Karol Życzkowski and Marco Piani for encouraging comments, and Mike McDonell for illuminating discussions. This work was supported by the Canada Excellence Research Chairs Program, the Natural Sciences and Engineering Research Council of Canada, and the UK EPSRC.

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