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Electromagnetic momenta and forces in dispersive dielectric media

Douglas H. Bradshaw [a], Zhimin Shi [b], Robert W. Boyd [b], Peter W. Milonni [c,*]

[a] Los Alamos National Laboratory, Los Alamos, NM 87545, USA
[b] The Institute of Optics, University of Rochester, Rochester, NY 14627, USA
[c] 104 Sierra Vista Dr, White Rock, NM 87544, USA

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ABSTRACT

When the effects of dispersion are included, neither the Abraham nor the Minkowski expression for electromagnetic momentum in a dielectric medium gives the correct recoil momentum for absorbers or emitters of radiation. The total momentum density associated with a field in a dielectric medium has three contributions: (i) the Abraham momentum density of the field, (ii) the momentum density associated with the Abraham force, and (iii) a momentum density arising from the dispersive part of the response of the medium to the field, the latter having a form evidently first derived by Nelson (1991) [8]. All three contributions are required for momentum conservation in the recoil of an absorber or emitter in a dielectric medium. We consider the momentum exchanged and the force on a polarizable particle (e.g., an atom or a small dielectric sphere) in a host dielectric when a pulse of light is incident upon it, including the dispersion of the dielectric medium as well as a dispersive component in the response of the particle to the field. The force can be greatly increased in slow-light dielectric media.

1. Introduction

Electromagnetic momentum in a dielectric medium is a subject with a very extensive literature, especially in connection with its different formulations. The two most favored forms by far are those of Abraham and Minkowski; as aptly remarked in a recent paper [1], "There is ...a bewildering array of experimental studies and associated theoretical analyses which appear to favor one or other of these momenta or, indeed, others." An aspect of this subject that has received surprisingly little attention concerns the effects of dispersion on the Minkowski and Abraham momenta and on the electromagnetic forces on polarizable particles. The intent of the present paper is to address such effects, which might help to clarify the physical interpretation of the Abraham and Minkowski momenta and the distinction between them.

We first review briefly the Abraham and Minkowski momenta for the situation usually considered—a dielectric medium assumed to be dispersionless and non-absorbing at a frequency ω. The Abraham and Minkowski momentum densities are, respectively,

$$\mathbf{P}_A = \frac{1}{c^2}\mathbf{E}\times\mathbf{H} \quad \text{and} \quad \mathbf{P}_M = \mathbf{D}\times\mathbf{B} \tag{1}$$

in the standard notation for the fields on the right-hand sides. We will take the permeability μ to be equal to its vacuum value μ_0, which is generally an excellent approximation at optical frequencies. For single photons the magnitudes of the Abraham and Minkowski momenta are given by (see Section 2)

$$p_A = \frac{1}{n}\frac{\hbar\omega}{c} \quad \text{and} \quad p_M = n\frac{\hbar\omega}{c}, \tag{2}$$

where n is the refractive index at frequency ω. From $\mathbf{D} = \epsilon_0 n^2\mathbf{E}$ it follows that

$$\frac{\partial\mathbf{P}_M}{\partial t} = \frac{\partial\mathbf{P}_A}{\partial t} + \mathbf{f}^A, \tag{3}$$

where

$$\mathbf{f}^A = \frac{1}{c^2}(n^2-1)\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}) \tag{4}$$

is the *Abraham force density*. For single-photon fields the momentum p^A associated with the Abraham force is $[(n^2-1)/n]\hbar\omega/c$, and (3) becomes $p_M = p_A + p^A$.

The Abraham momentum is generally regarded as the correct momentum of the electromagnetic field [2], whereas the Minkowski momentum evidently includes the momentum of the dielectric medium as well as that of the field. Ginzburg [3] calls p_M the momentum of a "photon in a medium," and notes that its use, together with energy and momentum conservation laws, yields correct results for Cerenkov radiation as well as the Doppler shift. Experiments appear by and large to indicate that it is the momentum $n\hbar\omega/c$ per photon that provides the recoil and radiation pressure experienced by an object immersed in a dielectric medium [4]. However, when dispersion ($dn/d\omega$) is accounted for, $n\hbar\omega/c$ is not

* Corresponding author.
E-mail address: pmil855242@att.net (P.W. Milonni).

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the Minkowski momentum of a photon, as we review in the following section.

This paper is organized as follows. In the following section we briefly discuss the generalization of the Abraham and Minkowski momenta to the case of a dispersive dielectric medium [5] and consider two examples: (i) the Doppler shift in a dielectric medium [6] and (ii) the displacement of a dielectric block on a frictionless surface due to the passage of a single-photon field through it [7]. A consistent description of momentum transfer in these examples requires that we account for momentum imparted to the medium. In Section 3 we calculate the force exerted by a quasimonochromatic plane wave on a polarizable particle and on a dispersive dielectric medium modeled as a continuum, and obtain a dispersive contribution to the latter in agreement with an expression that, to the best of our knowledge, was first derived, in a rather different way, by Nelson [8]. In Section 4 we consider the momentum exchange between a plane-wave pulse and an electrically polarizable particle immersed in a non-absorbing dielectric medium, and show that this momentum depends on both the dispersion of the medium and the variation with frequency of the polarizability; in particular, in slow-light media it can be large and in the direction opposite to that in which the field propagates. Section 5 presents derivations of some results relevant to Section 6, where we generalize the results of Section 4 to include absorption and discuss the forces exerted by a pulse on a small dielectric sphere in a host slow-light medium. Section 7 briefly summarizes our conclusions.

2. Abraham and Minkowski momenta for dispersive media

We first recall the expression for the total cycle-averaged energy density when a plane-wave monochromatic field [$\mathbf{E} = \mathbf{E}_\omega e^{-i\omega t}$, $\mathbf{H} = \mathbf{H}_\omega e^{-i\omega t}$, $\mathbf{H}_\omega^2 = (\epsilon/\mu_0)\mathbf{E}_\omega^2$] propagates in a dispersive dielectric at a frequency ω at which absorption is negligible [9]:

$$u = \frac{1}{4}\left[\frac{d}{d\omega}(\epsilon\omega)\mathbf{E}_\omega^2 + \mu_0\mathbf{H}_\omega^2\right], \tag{5}$$

or equivalently, in terms of $\mathbf{E}_\omega$ and the group index $n_g = d(n\omega)/d\omega$,

$$u = \frac{1}{2}\epsilon_0 n n_g \mathbf{E}_\omega^2. \tag{6}$$

When the field is quantized in a volume V, u is in effect replaced by $q\hbar\omega/V$, where q is the expectation value of the photon number in the volume V; therefore, from Eq. (6), $\mathbf{E}_\omega^2$ is effectively $2\hbar\omega/(\epsilon_0 n n_g V)$ per photon. Thus, for single photons, the Abraham momentum defined by Eq. (1) is

$$p_A = \frac{n}{c}\frac{1}{2}\epsilon_0\frac{2\hbar\omega}{\epsilon_0 n n_g V}V = \frac{1}{n_g}\frac{\hbar\omega}{c}. \tag{7}$$

Similarly,

$$p_M = \frac{n^2}{n_g}\frac{\hbar\omega}{c}, \tag{8}$$

which follows from the definition in Eq. (1) and the relation $\mathbf{D} = \epsilon_0 n^2\mathbf{E}$; thus $p_M = n^2 p_A$. These same expressions for p_A and p_M can of course be obtained more formally by quantizing the fields $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, and $\mathbf{B}$ in a dispersive medium [5].

Two examples serve to clarify the differences among the momenta involved in the momentum exchange between light and matter. The first example is based on an argument of Fermi's that the Doppler effect is a consequence of this momentum exchange [6], as follows. Consider an atom of mass M inside a host dielectric medium with refractive index $n(\omega)$. The atom has a sharply defined transition frequency ω_0 and is initially moving with velocity v away from a source of light of frequency ω. Because the light in the atom's reference frame has a Doppler-shifted frequency $\omega(1 - nv/c)$ determined by the phase velocity (c/n) of light in the medium, the atom can absorb a photon if $\omega(1 - nv/c) = \omega_0$, or if

$$\omega \cong \omega_0(1 + nv/c). \tag{9}$$

We denote the momentum associated with a photon in the medium by $\wp$ and consider the implications of (non-relativistic) energy and momentum conservation. The initial energy is $E_i = \hbar\omega + \frac{1}{2}Mv^2$, and the final energy, after the atom has absorbed a photon, is $\frac{1}{2}Mv'^2 + \hbar\omega_0$, where v' is the velocity of the atom after absorption. The initial momentum is $\wp + Mv$, and the final momentum is just Mv'. Therefore

$$\frac{1}{2}M(v'^2 - v^2) \cong Mv(v' - v) = Mv(\wp/M) = \hbar(\omega - \omega_0), \tag{10}$$

or $\omega \cong \omega_0 + \wp v/\hbar$. From Eq. (9) and $\omega \cong \omega_0$ we conclude that

$$\wp = n\frac{\hbar\omega}{c}. \tag{11}$$

Thus, once we accept the fact that the Doppler shift depends on the refractive index of the medium according to Eq. (9), we are led by energy and momentum conservation to conclude that an atom in the medium must recoil with momentum (11) when it absorbs (or emits) a photon of energy $\hbar\omega$. Momentum conservation in this example is discussed in more detail below.

In our second example we consider, following Balazs [7], a rigid block of mass M, refractive index n, and length a, initially sitting at rest on a frictionless surface. A single-photon pulse of frequency ω passes through the block, which is assumed to be non-absorbing at frequency ω and to have anti-reflection coatings on its front and back surfaces. The length a of the block is presumed to be much larger than the length of the pulse. If the photon momentum is $\wp_{\text{in}}$ inside the block and $\wp_{\text{out}}$ outside, the block picks up a momentum $MV = \wp_{\text{out}} - \wp_{\text{in}}$ when the pulse enters. If the space outside the block is vacuum, $\wp_{\text{out}} = mc$, where $m = E/c^2 = \hbar\omega/c^2$. Similarly $\wp_{\text{in}} = mv_p$, where v_p is the velocity of light in the block. Without dispersion, $v_p = c/n$ and the momentum of the photon in the block is evidently $\wp_{\text{in}} = mc/n = \hbar\omega/nc$. The effect of dispersion is to replace $v_p = c/n$ by $v_g = c/n_g$ and $\wp_{\text{in}} = \hbar\omega/nc$ by $\wp_{\text{in}} = \hbar\omega/n_g c$. With or without dispersion, this example suggests that the photon momentum in the medium has the Abraham form. Note that the essential feature of Balazs's argument is simply that the velocity of light in the medium is v_p (or, more generally, v_g). This, together with momentum conservation, is what leads him to conclude that the momentum of the field has the Abraham form.

This prediction can in principle be tested experimentally. Conservation of momentum requires, according to Balazs's argument, that $MV = m(c - v_g)$. When the pulse exits, the block recoils and comes to rest, and is left with a net displacement

$$\Delta x = V\Delta t = \frac{m}{M}(c - v_g)\frac{a}{v_g} = \frac{\hbar\omega}{Mc^2}(n_g - 1)a \tag{12}$$

as a result of the light having passed through it. This is the prediction for the net displacement based on the momentum p_A given in (7). If the photon momentum inside the block were assumed to have the Minkowski form $n^2\hbar\omega/cn_g$ given in (8), however, the displacement of the block would in similar fashion be predicted to be

$$\Delta x = \frac{\hbar\omega}{Mc^2}a(n_g - n^2), \tag{13}$$

and if it were assumed to be $n\hbar\omega/c$, as in Eq. (11), the prediction would be that the net displacement of the block is

$$\Delta x = \frac{\hbar\omega}{Mc^2}an_g(1 - n). \tag{14}$$

These different assumptions about the photon momentum can lead to different predictions not only for the magnitude of the block displacement but also for its direction.

The first (Doppler) example suggests at first thought that the momentum of the photon is $n\hbar\omega/c$ [Eq. (11)], while the second (Balazs) example indicates that it is $\hbar\omega/n_g c$. Let us consider more carefully the first example. There is ample experimental evidence that the Doppler shift is $n v\omega/c$ regardless of dispersion, as we have assumed, but does this imply that the momentum of a photon in a dielectric is in fact $n\hbar\omega/c$? We will show in the following section that the forces exerted by a plane monochromatic wave on the polarizable particles of a dielectric result in a momentum density of magnitude

$$p_{\rm med} = \frac{\epsilon_0}{2c} n(nn_g - 1)E_\omega^2 = (n - \frac{1}{n_g})\frac{\hbar\omega}{c}\frac{1}{V}; \tag{15}$$

the second equality applies to a single photon, and follows from the replacement of $\mathbf{E}_\omega^2$ by $2\hbar\omega/(\epsilon_0 nn_g V)$, as discussed earlier. Now from the fact that the Doppler shift implies that an absorber (or emitter) inside a dielectric recoils with momentum $n\hbar\omega/c$, all we can safely conclude from momentum conservation is that a momentum $n\hbar\omega/c$ is taken from (or given to) the *combined system of field and dielectric*. Given that the medium has a momentum density (15) due to the force exerted on it by the propagating field, we can attribute to the field (by conservation of momentum) a momentum density

$$n\frac{\hbar\omega}{c}\frac{1}{V} - P_{\rm med} = \frac{1}{n_g}\frac{\hbar\omega}{c}\frac{1}{V} = p_A. \tag{16}$$

That is, the momentum of the field in this interpretation is given by the Abraham formula, consistent with the conclusion of the Balazs thought experiment. The recoil momentum $n\hbar\omega/c$, which in general differs from both the Abraham and the Minkowski momenta, evidently gives the momentum not of the field as such but of the combined system of field plus dielectric. It is the momentum density equal to the *total* energy density $u = \hbar\omega/V$ for a monochromatic field divided by the phase velocity c/n of the propagating wave. As already mentioned, experiments on the recoil of objects immersed in dielectric media have generally indicated that the recoil momentum is $n\hbar\omega/c$ per unit of energy $\hbar\omega$ of the field, just as in the Doppler effect. But this should not be taken to mean that $n\hbar\omega/c$ is the momentum of a "photon" existing independently of the medium in which the field propagates. Regardless of how this momentum is apportioned between the field and the medium in which it propagates, the important thing for the theory, of course, is that it correctly predicts the *observable forces* exerted by electromagnetic fields. We next turn our attention specifically to the forces acting on polarizable particles in applied electromagnetic fields.

3. Momenta and forces on polarizable particles

We will make the electric dipole approximation and consider field frequencies such that absorption is negligible. Then the induced electric dipole moment of a particle in a field of frequency ω is $\mathbf{d} = \alpha(\omega)\mathbf{E}_\omega \exp(-i\omega t)$, and the polarizability $\alpha(\omega)$ may be taken to be real. With these assumptions we now consider the forces acting on such particles in applied, quasi-monochromatic fields.

We begin with the Lorentz force on an electric dipole moment $\mathbf{d}$ in an electromagnetic field [10]:

$$\mathbf{F} = (\mathbf{d}\cdot\nabla)\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B} = (\mathbf{d}\cdot\nabla)\mathbf{E} + \mathbf{d}\times(\nabla\times\mathbf{E}) + \frac{\partial}{\partial t}(\mathbf{d}\times\mathbf{B}) \equiv \mathbf{F}_E + \mathbf{F}_B, \tag{17}$$

where we define

$$\mathbf{F}_E = (\mathbf{d}\cdot\nabla)\mathbf{E} + \mathbf{d}\times(\nabla\times\mathbf{E}), \tag{18}$$

$$\mathbf{F}_B = \frac{\partial}{\partial t}(\mathbf{d}\times\mathbf{B}). \tag{19}$$

In writing the second equality in (17) we have used the Maxwell equation $\partial\mathbf{B}/\partial t = -\nabla\times\mathbf{E}$. The dipole moment of interest here is induced by the electric field. Writing

$$\mathbf{E} = \mathscr{E}_0(\mathbf{r},t)e^{-i\omega t} = e^{-i\omega t}\int_{-\infty}^{\infty} d\Delta\tilde{\mathscr{E}}_0(\mathbf{r},\Delta)e^{-i\Delta t}, \tag{20}$$

in which $|\partial\mathscr{E}_0/\partial t| \ll \omega|\mathscr{E}_0|$ for a quasi-monochromatic field, we approximate $\mathbf{d}$ as follows:

$$\begin{aligned}\mathbf{d}(\mathbf{r},t) &= \int_{-\infty}^{\infty} d\Delta\alpha(\omega+\Delta)\tilde{\mathscr{E}}_0(\mathbf{r},\Delta)e^{-i(\omega+\Delta)t} \\ &\cong \int_{-\infty}^{\infty} d\Delta[\alpha(\omega)+\Delta\alpha'(\omega)]\tilde{\mathscr{E}}_0(\mathbf{r},\Delta)e^{-i(\omega+\Delta)t} \\ &= \left[\alpha(\omega)\mathscr{E}_0(\mathbf{r},t) + i\alpha'(\omega)\frac{\partial\mathscr{E}_0}{\partial t}\right]e^{-i\omega t}.\end{aligned} \tag{21}$$

Here $\alpha\prime = d\alpha/d\omega$ and we assume that higher-order dispersion is sufficiently weak that terms $d^m\alpha/d\omega^m$ can be neglected for $m \geqslant 2$. Putting (21) into (18), we obtain after some straightforward manipulations and cycle-averaging the force

$$\mathbf{F}_E = \nabla\left[\frac{1}{4}\alpha(\omega)|\mathscr{E}|^2\right] + \frac{1}{4}\alpha'(\omega)\mathbf{k}\frac{\partial}{\partial t}|\mathscr{E}|^2, \tag{22}$$

where $\mathscr{E}$ and $\mathbf{k}$ are defined by writing $\mathscr{E}_0(\mathbf{r},t) = \mathscr{E}(\mathbf{r},t)e^{i\mathbf{k}\cdot\mathbf{r}}$. Since the refractive index n of a medium in which local field corrections are negligible is given in terms of α by $n^2 - 1 = N\alpha/\epsilon_0$, N being the density of dipoles in the dielectric, we have $\alpha\prime = (2n\epsilon_0/N)(dn/d\omega)$ and

$$\mathbf{F}_E = \nabla\left[\frac{1}{4}\alpha(\omega)|\mathscr{E}|^2\right] + \frac{\epsilon_0}{2N}\mathbf{k}n\frac{dn}{d\omega}\frac{\partial}{\partial t}|\mathscr{E}|^2. \tag{23}$$

The first term is the "dipole force" associated with the energy $W = -\frac{1}{2}\alpha(\omega)\mathbf{E}^2$ involved in inducing an electric dipole moment in an electric field:

$$W = -\int_0^{\mathbf{E}}\mathbf{d}\cdot d\mathbf{E} = -\alpha(\omega)\int_0^{\mathbf{E}}\mathbf{E}\cdot d\mathbf{E} = -\frac{1}{2}\alpha(\omega)\mathbf{E}^2. \tag{24}$$

The second term in (23) is non-vanishing only because of dispersion ($dn/d\omega \neq 0$). It is in the direction of propagation of the field, and implies for a uniform density N of atoms per unit volume a momentum density of magnitude

$$P_D = \frac{1}{2}\epsilon_0 n^2\frac{dn}{d\omega}\frac{\omega}{c}|\mathscr{E}|^2 = \frac{1}{2}\frac{\epsilon_0}{c}n^2(n_g - n)|\mathscr{E}|^2, \tag{25}$$

since $k = n(\omega)\omega/c$. This momentum density comes specifically from the dispersion ($dn/d\omega$) of the medium.

The force $\mathbf{F}_B$ defined by (19), similarly, implies a momentum density $\mathbf{P}^A$ imparted to the medium:

$$\mathbf{P}^A = N\mathbf{d}\times\mathbf{B}. \tag{26}$$

As the notation suggests, this momentum density is associated with the Abraham force density (4). The result of a straightforward evaluation of $\mathbf{P}^A$ based on (21) and $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ is

$$\mathbf{P}^A = \frac{1}{2}\epsilon_0(n^2-1)\frac{\mathbf{k}}{\omega}|\mathscr{E}|^2, \quad P^A = \frac{1}{2}\frac{\epsilon_0}{c}n(n^2-1)|\mathscr{E}|^2, \tag{27}$$

when we use $\mathbf{k}\cdot\mathbf{E} = 0$ and our assumption that $|\dot{\mathscr{E}}_0| \ll \omega|\mathscr{E}_0|$. The magnitude of the total momentum density in the medium due to the force of the field on the dipoles is therefore

$$\begin{aligned}P_{\rm med} = P_D + P^A &= \frac{\epsilon_0}{2c}\left[n^2(n_g - n) + n(n^2-1)\right]|\mathscr{E}|^2 \\ &= \frac{\epsilon_0}{2c}n(nn_g - 1)|\mathscr{E}|^2\end{aligned} \tag{28}$$

in the approximation in which the field is sufficiently uniform that we can ignore the dipole force $\nabla\left[\frac{1}{4}\alpha|\mathscr{E}|^2\right]$.

The complete momentum density for the field and the medium is obtained by adding to (28) the Abraham momentum density P_A of the field. According to (1), $P_A = (\epsilon_0/2c)n|\mathscr{E}|^2$, and so the total momentum density is

$$P_A + P_D + P^A = \frac{\epsilon_0}{2c}\left[n + n(nn_g - 1)\right]|\mathscr{E}|^2 = \frac{\epsilon_0}{2c}n^2 n_g|\mathscr{E}|^2 \tag{29}$$

if the dipole force is negligible. To express these results in terms of single photons, we again replace $|\mathscr{E}_0|^2$ by $2\hbar\omega/(\epsilon_0 nn_g V)$; then (29) takes the form

$$p_A + p_D + p^A = n\frac{\hbar\omega}{c}\frac{1}{V}, \tag{30}$$

consistent with the discussion in the preceding section. This is the total momentum density per photon, assuming that the dipole force is negligible. The momentum density of the medium per photon follows from (28):

$$p_{\rm med} = p_D + p^A = \frac{\epsilon_0}{2c}n(nn_g - 1)\frac{2\hbar\omega}{nn_g\epsilon_0 V} = (n - \frac{1}{n_g})\frac{\hbar\omega}{c}\frac{1}{V}, \tag{31}$$

as stated earlier (Eq. (15)).

Consider the example of spontaneous emission by a guest atom in a host dielectric medium. The atom loses energy $\hbar\omega_0$, and the quantum (photon in the medium) of excitation carries away from the atom not only this energy but also a linear momentum $n\hbar\omega/c$ (Eq. (30)). The atom therefore recoils with momentum $n\hbar\omega/c$ [11].

The momentum density (25) was obtained by Nelson [8] in a rigorous treatment of a deformable dielectric based on a Lagrangian formulation; in the present paper a dielectric medium is treated as an idealized rigid body. From a microscopic perspective, this part of the momentum density of the medium is attributable directly to the second term on the right-hand side of (21), i.e., to the part of the induced dipole moment that arises from dispersion. In the Appendix the relation of this term to the formula (5) for the total energy density is reviewed; the term is obviously a general property of induced dipole moments in applied fields. Consider, for example, a two-level atom driven by a quasi-monochromatic field with frequency ω far-detuned from the atom's resonance frequency ω_0. In the standard u, v notation for the off-diagonal components of the density matrix in the rotating-wave approximation [12],

$$u(t) - iv(t) \cong \frac{1}{\Delta}\chi(t) + \frac{i}{\Delta^2}\frac{\partial\chi}{\partial t} + \cdots , \tag{32}$$

where $\chi(t)$ is the Rabi frequency and Δ is the detuning. The polarizability is proportional to $1/\Delta$ in this approximation, and therefore (32) is just a special case of (21).

4. Momentum exchange between a light pulse and an induced dipole

We next consider the momentum exchange between a plane-wave *pulse* and a single polarizable particle. We will assume again that the particle is characterized by a real polarizability $\alpha(\omega)$ and that it is surrounded by a host medium with refractive index $n_b(\omega)$. The electric field is assumed to be

$$\mathbf{E}(z,t) = \mathscr{E}(t - z/v_{bg})\cos(\omega t - kz), \tag{33}$$

with $k = n_b(\omega)\omega/c$ and group velocity $v_{bg} = c/n_{bg}$, $n_{bg} = (d/d\omega)(\omega n_b)$.

The force acting on the particle is $\mathbf{F}_E + \mathbf{F}_B$. $\mathbf{F}_B$ reduces to $\frac{1}{2}\alpha(\omega)(\mathbf{k}/\omega)(\partial/\partial t)|\mathscr{E}|^2$, obtained by multiplying (27) by a volume V describing the pulse, replacing $n^2 - 1$ by $N\alpha/\epsilon_0$ with $NV = 1$ for the single particle, and differentiation with respect to time. $\mathbf{F}_E$ follows from (22). Then the force acting on the particle is in the z direction and has the (cycle-averaged) magnitude

$$F = \frac{1}{4}\alpha(\omega)\frac{\partial}{\partial z}\mathscr{E}^2 + \frac{1}{4}\alpha'(\omega)n_b(\omega)\frac{\omega}{c}\frac{\partial}{\partial t}\mathscr{E}^2 + \frac{1}{2c}\alpha(\omega)n_b(\omega) \times \frac{\partial}{\partial t}\mathscr{E}^2, \tag{34}$$

where now we retain the dipole force, given by the first term on the right-hand side. The momentum of the particle at z at time T is

$$\begin{aligned} p &= \int_{-\infty}^{T} F dt \\ &= \frac{1}{4}\alpha\int_{-\infty}^{T}\frac{\partial}{\partial z}\mathscr{E}^2(t - z/v_{bg})dt + \frac{1}{4c}\alpha' n_b\omega\int_{-\infty}^{T}\frac{\partial}{\partial t}\mathscr{E}^2(t - z/v_{bg})dt + \frac{1}{2c}\alpha n_b\int_{-\infty}^{T}\frac{\partial}{\partial t}\mathscr{E}^2(t - z/v_{bg})dt \\ &= -\frac{1}{4}\alpha\frac{1}{v_{bg}}\mathscr{E}^2 + \frac{n_b}{4c}\alpha'\omega\mathscr{E}^2 + \frac{1}{2}\alpha\frac{n_b}{c}\mathscr{E}^2 \\ &= \frac{1}{4c}\left[(2n_b - n_{bg})\alpha + n_b\omega\alpha'\right]\mathscr{E}^2(T - z/v_{bg}). \end{aligned} \tag{35}$$

Hinds and Barnett [1] have considered the force on a two-level atom due to a pulse of light in free space. In this case $n_b = n_{bg} = 1$ and (35) reduces to

$$p = \frac{1}{4c}[\alpha + \omega\alpha']\mathscr{E}^2. \tag{36}$$

Following Hinds and Barnett, we argue that a pulse occupying the volume V in the neighborhood of the atom in free space corresponds to a number $q = \frac{1}{2}\epsilon_0\mathscr{E}^2 V/\hbar\omega$ of photons, so that

$$p = \frac{1}{2c}[\alpha + \omega\alpha']\frac{\hbar\omega}{\epsilon_0 V}q. \tag{37}$$

$\alpha = \epsilon_0(n^2 - 1)/N$, where n is the refractive index in the case of N polarizable particles per unit volume. Then

$$\begin{aligned} p &= \frac{1}{2c}\left[\frac{\epsilon_0(n^2 - 1)}{N} + \frac{2\epsilon_0 n}{N}\omega\frac{dn}{d\omega}\right]\frac{\hbar\omega}{c}q \\ &\cong \left[n - 1 + \omega\frac{dn}{d\omega}\right]\frac{\hbar\omega}{c}q \equiv K\frac{\hbar\omega}{c}q. \end{aligned} \tag{38}$$

This is the momentum imparted to the particle, which implies a change in *field* momentum per photon equal to

$$\frac{\hbar\omega}{c}[1 - K] \cong \frac{\hbar\omega}{c}\frac{1}{1 + K} = \frac{\hbar\omega}{n_g c} \tag{39}$$

if $|K| \ll 1$, where $n_g = (d/d\omega)(n\omega)$. As in the case of a two-level atom considered by Hinds and Barnett, this corresponds to the Abraham momentum; our result simply generalizes theirs in replacing n by n_g in the expression for the change in photon momentum.

In the case of a polarizable particle in a host dielectric rather than in free space we obtain, from (35),

$$p = \frac{I}{2\epsilon_0 c^2}\left[(2 - \frac{n_{bg}}{n_b})\alpha + \omega\alpha'\right], \tag{40}$$

where the intensity $I = (1/2)c\epsilon_0 n_b\mathscr{E}^2$. If dispersion in the medium and in the polarizability of the guest particle are negligible, we can set $n_{bg} = n$ and $\alpha\prime = 0$, and then (40) reduces to a well-known expression [13]. However, this momentum can be large in a slow-light medium (n_{bg} large), for example, because the gradient of the field (33) responsible for the dipole force on the particle is large [14]; this is a consequence of the spatial compression of a pulse in a slow-light medium. We discuss this case further in Section 6.

But first we return to some other well-known results that are relevant there.

5. Electric dipole radiation rate and Rayleigh scattering [15]

A Hertz vector $\mathbf{\Pi}(\mathbf{r},\omega)$ can be defined for a dielectric medium, analogous to the case of free space [16], by writing the electric and magnetic field components at frequency ω as

$$\mathbf{E}(\mathbf{r},\omega) = k_0^2[\epsilon_b(\omega)/\epsilon_0]\mathbf{\Pi}(\mathbf{r},\omega) + \nabla[\nabla\cdot\mathbf{\Pi}(\mathbf{r},\omega)], \tag{41}$$

$$\mathbf{H}(\mathbf{r},\omega) = -i\omega\epsilon_b(\omega)\nabla\times\mathbf{\Pi}(\mathbf{r},\omega). \tag{42}$$

Here $k_0 = \omega/c$ and we denote by $\epsilon_b(\omega)$ the (real) permittivity of the dielectric. We will be interested here in a dipole source inside the "background" dielectric medium. The identifications (41) and (42) are consistent with the propagation of a wave of frequency ω with the phase velocity $c/n_b(\omega)$ in the medium $\left[n_b(\omega) = \sqrt{\epsilon_b(\omega)/\epsilon_0}\right]$, as will be clear in the following.

The curl of $\mathbf{E}(\mathbf{r},\omega)$ in (41) is simply

$$\nabla\times\mathbf{E}(\mathbf{r},\omega) = k_0^2[\epsilon_b(\omega)/\epsilon_0]\nabla\times\mathbf{\Pi}(\mathbf{r},\omega), \tag{43}$$

since the curl of a gradient is zero. Now apply the curl operation to this equation, assuming no free currents and therefore $\nabla\times\mathbf{H}(\mathbf{r},\omega) = -i\omega\mathbf{D}(\mathbf{r},\omega)$:

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{E}) &= i\omega\mu_0\nabla\times\mathbf{H} = \omega^2\mu_0\mathbf{D}\\ &= k_0^2[\epsilon_b(\omega)/\epsilon_0]\nabla\times(\nabla\times\mathbf{\Pi})\\ &= k_0^2[\epsilon_b(\omega)/\epsilon_0]\left[\nabla(\nabla\cdot\mathbf{\Pi}) - \nabla^2\mathbf{\Pi}\right],\end{aligned} \tag{44}$$

implying

$$\nabla^2\mathbf{\Pi} = \frac{\epsilon_0}{\epsilon_b}\frac{\omega^2}{k_0^2}\mu_0\mathbf{D} + \nabla(\nabla\cdot\mathbf{\Pi}) = -\frac{1}{\epsilon_b}\mathbf{D} + \left[\mathbf{E} - \frac{\epsilon_b}{\epsilon_0}k_0^2\mathbf{\Pi}\right], \tag{45}$$

$$\nabla^2\mathbf{\Pi} + k^2\mathbf{\Pi} = \mathbf{E} - \frac{1}{\epsilon_b}\mathbf{D}, \quad k^2 = k_0^2\epsilon_b(\omega)/\epsilon_0 = n_b^2(\omega)\omega^2/c^2. \tag{46}$$

If $\mathbf{D}(\mathbf{r},\omega) = \epsilon_b(\omega)\mathbf{E}(\mathbf{r},\omega)$, the right-hand side is zero, and all we have done is rederived what we already know: the field propagates with phase velocity $\omega/k(\omega) = c/n_b(\omega)$. Suppose, however, that within the medium there is a localized source characterized by a dipole moment density $\mathbf{P}_s(\mathbf{r},\omega) = \mathbf{p}_0(\omega)\delta^3(\mathbf{r})$. Then $\mathbf{D} = \epsilon_b\mathbf{E} + \mathbf{P}_s$ and

$$\nabla^2\mathbf{\Pi} + k^2\mathbf{\Pi} = -\frac{1}{\epsilon_b}\mathbf{p}_0(\omega)\delta^3(\mathbf{r}). \tag{47}$$

The solution of this equation for $\mathbf{\Pi}(\mathbf{r},\omega)$ is simply

$$\mathbf{\Pi}(\mathbf{r},\omega) = \frac{1}{4\pi\epsilon_b(\omega)}\mathbf{p}_0(\omega)\frac{e^{ikr}}{r}, \tag{48}$$

and from this one obtains the electric and magnetic fields due to the source in the medium. In the far field, assuming $\mathbf{p}_0 = p\hat{z}$ and letting θ be the angle between the z axis and the observation point,

$$E_\theta = \frac{k_0^2 p}{4\pi\epsilon_0}\sin\theta\frac{e^{ikr}}{r}, \tag{49}$$

$$H_\phi = \frac{n_b k_0^2 p}{4\pi\epsilon_0}\sqrt{\frac{\epsilon_0}{\mu_0}}\sin\theta\frac{e^{ikr}}{r}, \tag{50}$$

in spherical coordinates. The Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ implies the radiation rate

$$P = \frac{n_b p^2\omega^4}{12\pi\epsilon_0 c^3}, \tag{51}$$

analogous to the fact that the spontaneous emission rate of an atom in a dielectric without local field corrections is proportional to the (real) refractive index at the emission frequency.

5.1. Polarizability of a dielectric sphere

Suppose, somewhat more generally, that the source within the medium occupies a volume V and is characterized by a permittivity $\epsilon_s(\omega)$. Then $\mathbf{D}(\mathbf{r},\omega) = \epsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega)$, where $\epsilon = \epsilon_s(\omega)$ within the volume V occupied by the source and $\epsilon(\mathbf{r},\omega) = \epsilon_b(\omega)$ outside this volume, and

$$\nabla^2\mathbf{\Pi} + k^2\mathbf{\Pi} = [1 - \epsilon(\mathbf{r},\omega)/\epsilon_b(\omega)]\mathbf{E}. \tag{52}$$

The solution of this equation is

$$\mathbf{\Pi}(\mathbf{r},\omega) = -\frac{1}{4\pi}\left[1 - \frac{\epsilon_s(\omega)}{\epsilon_b(\omega)}\right]\int_V d^3r'\mathbf{E}(\mathbf{r}',\omega)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{53}$$

Suppose further that the extent of the volume V is sufficiently small compared to a wavelength that we can approximate (53) by

$$\mathbf{\Pi}(\mathbf{r},\omega) = -\frac{1}{4\pi}\left[1 - \frac{\epsilon_s(\omega)}{\epsilon_b(\omega)}\right]V\mathbf{E}_{\text{ins}}(\omega)\frac{e^{ikr}}{r}, \tag{54}$$

with r the distance from the center of the source (at $\mathbf{r} = 0$) to the observation point and $\mathbf{E}_{\text{ins}}(\omega)$ the (approximately constant) electric field in the source volume V. This has the same form as (48) with $\mathbf{p}_0(\omega) = \epsilon_b(\omega)[\epsilon_s(\omega)/\epsilon_b(\omega) - 1]V\mathbf{E}_{\text{ins}}(\omega)$. In other words, $\mathbf{\Pi}(\mathbf{r},\omega)$ has the same form as the Hertz vector for an electric dipole moment

$$\mathbf{p}_0(\omega) = [\epsilon_s(\omega) - \epsilon_b(\omega)]V\mathbf{E}_{\text{ins}}(\omega). \tag{55}$$

Consider, for example, a small dielectric sphere of radius a: $V = 4\pi a^3/3$. The field inside such a sphere is $\mathbf{E}_{\text{ins}}(\omega) = [3\epsilon_b/(\epsilon_s + 2\epsilon_b)]\mathbf{E}_b(\omega)$, where $\mathbf{E}_b(\omega)$ is the (uniform) electric field in the medium in the absence of the source. The dipole moment (55) in this case is therefore related to the external field $\mathbf{E}_{\text{out}}(\omega)$ by $\mathbf{p}_0(\omega) = \alpha(\omega)\mathbf{E}_{\text{out}}(\omega)$, where the polarizability

$$\alpha(\omega) = 4\pi\epsilon_b\left(\frac{\epsilon_s - \epsilon_b}{\epsilon_s + 2\epsilon_b}\right)a^3. \tag{56}$$

5.2. Rayleigh attenuation coefficient

The cross-section for Rayleigh scattering for an ideal gas of refractive index $n(\omega)$ can be deduced as follows [17]. An electric field $\mathbf{E}_0\cos\omega t$ induces an electric dipole moment $\mathbf{p}(t) = \alpha(\omega)\mathbf{E}_0\cos\omega t$ in each of N isotropic, polarizable particles per unit volume, each particle having a spatial extent small compared to a wavelength. The power radiated by this dipole is, from Eq. (51),

$$\frac{dW_{\text{rad}}}{dt} = n(\omega)\frac{\omega^4}{12\pi\epsilon_0 c^3}\alpha^2(\omega)\mathbf{E}_0^2 \equiv \sigma_{\text{R}}(\omega)I, \tag{57}$$

where W_{rad} denotes energy of the radiated field, $I = \frac{1}{2}n(\omega)c\epsilon_0\mathbf{E}_0^2$ is the intensity of the field incident on the dipole, and

$$\sigma_{\text{R}}(\omega) = \frac{1}{6\pi N^2}\left(\frac{\omega}{c}\right)^4\left[n^2(\omega) - 1\right]^2 \tag{58}$$

is the (Rayleigh) scattering cross-section. We have assumed that local field corrections are negligible and used the formula $n^2(\omega) - 1 = N\alpha(\omega)/\epsilon_0$ to express $\sigma_{\text{R}}(\omega)$ in terms of the refractive index $n(\omega)$. The attenuation coefficient is then

$$a_R = N\sigma_{\text{R}} = \frac{1}{6\pi N}\left(\frac{\omega}{c}\right)^4\left[n^2(\omega) - 1\right]^2. \tag{59}$$

Rosenfeld [18] obtains instead

$$a_R = N\sigma_{\text{R}} = \frac{1}{6\pi n(\omega)N}\left(\frac{\omega}{c}\right)^4\left[n^2(\omega) - 1\right]^2 \tag{60}$$

because he does not account for the factor $n(\omega)$ in the dipole radiation rate (57). Rayleigh's derivation of (59) follows essentially the one just given, but the factor $n(\omega)$ appears in neither the dipole radiation rate nor the expression for the intensity (or actually, in his derivation, the energy density) [19]. In practice the difference between (59) and (60) is negligible for the case assumed here of a dilute medium [20].

6. Force on a dielectric sphere

The expression (34) for the force on a polarizable particle in a field (33) may be generalized to allow for absorption by the particle simply by taking the polarizability $\alpha(\omega)$ in (21) to be complex. Assuming again that $\mathscr{E}$ is slowly varying in time compared to $\exp(-i\omega t)$, and slowly varying in space compared to $\exp(ikz)$, we obtain

$$F = \frac{1}{4c}\left[(2n_b - n_{bg})\alpha_R + n_b\omega\alpha_R'\right]\frac{\partial}{\partial\tau}|\mathscr{E}|^2 + \frac{1}{2}n_b\frac{\omega}{c}\alpha_I|\mathscr{E}|^2, \tag{61}$$

where $\tau = t - n_{bg}z/c$ and α_R and α_I are the real and imaginary parts, respectively, of $\alpha(\omega)$. If we replace n_{bg} by n_b and take $\alpha_{R'} \cong 0$, we recover results that may be found in many previous works when absorption is assumed to be negligible [13]. The last term in (61) is the absorptive contribution to Eq. (7) of a paper by Chaumet and Nieto-Vesperinas [21] when the field is assumed to have the form (33).

The polarizability in the case of a dielectric sphere of radius a much smaller than the wavelength of the field is given by (56). Dispersion affects the force (61) both through the group index (n_{bg}) of the host dielectric medium and the variation of the real part of the sphere's polarizability with frequency ($\alpha_{R'}$). The latter depends on both the intrinsic frequency dependence of the permittivity of the material of the sphere and the frequency dependence of the refractive index of the host medium. If these dispersive contributions to the force exceed the remaining two contributions to the force (56),

$$F \cong \frac{1}{4c}\left[-\alpha_R n_{bg} + n_b\omega\alpha_R'\right]\frac{\partial}{\partial\tau}|\mathscr{E}|^2. \tag{62}$$

Using (56) for this case, we obtain

$$F \cong -\frac{3\pi\epsilon_0 a^3}{c}n_{bg}\frac{n_s^2 n_b^4}{(n_s^2+2n_b^2)^2}\frac{\partial}{\partial\tau}|\mathscr{E}|^2 \tag{63}$$

if the dispersion of the dielectric material constituting the sphere is much smaller than that of the host dielectric medium, i.e., if $d\epsilon_s/d\omega \ll d\epsilon_b/d\omega$. (Here n_s is the refractive index at frequency ω of the material of the sphere.) This result implies that, in the case of a slow-light host medium ($n_{bg} \gg 1$), the force on the sphere can be much larger than would be the case in a "normally dispersive" medium, and is in the direction opposite to that in which the field propagates.

The simple formula (63), and similar expressions obtained in other limiting cases of (61), obviously allow for a wide range of forces when a pulse of radiation is incident on a dielectric sphere in a host dielectric medium. Here we make only a few remarks concerning the last term in (61). Although we have associated this contribution to the force with absorption, such a force appears even if the sphere does not absorb any radiation of frequency ω. This is because there must be an imaginary part of the polarizability simply because the sphere scatters radiation and thereby takes energy out of the incident field. According to the optical theorem in this case of scattering by a non-absorbing polarizable particle that is small compared to the wavelength of the field, the imaginary part of the polarizability is related to the complete (complex) polarizability as follows [22]:

$$\alpha_I(\omega) = \frac{1}{4\pi\epsilon_0}\frac{2\omega^3}{3c^3}n_b|\alpha(\omega)|^2. \tag{64}$$

Then the force proportional to $\alpha_I(\omega)$ in (61) is

$$F_{\text{scat}} \equiv \frac{1}{2}n_b^5\frac{\omega}{c}\alpha_I|\mathscr{E}|^2 = \frac{8\pi}{3}\left(\frac{\omega}{c}\right)^4\frac{n_b^5 I}{c}\left(\frac{\epsilon_s-\epsilon_b}{\epsilon_s+2\epsilon_b}\right)^2 a^6, \tag{65}$$

which is just the well-known "scattering force" [23] on a dielectric sphere in a medium with refractive index n_b, which may be taken to be real in the approximation in which the field is far from any absorption resonances of the sphere.

7. Conclusions

In this attempt to better understand the different electromagnetic momenta and the forces on electrically polarizable particles in dispersive dielectric media, we have made several simplifications, including the neglect of any surface effects, the treatment of the medium as a non-deformable body, and the approximation of plane-wave fields. We have shown that conservation of momentum, even in seemingly simple examples such as the Doppler effect, generally requires consideration not only of the Abraham momentum and the Abraham force, but also of a contribution to the momentum of the medium due specifically to the dispersive nature of the medium. We have generalized some well-known expressions for the forces on particles immersed in a dielectric medium to include dispersion. While we have presented arguments in favor of the interpretation of the Abraham momentum as the momentum of the field, our simplified analyses lead us to the conclusion that neither the Abraham nor the Minkowski expressions for momentum give the recoil momentum of a particle in a dispersive dielectric medium. Finally we have shown that the force exerted on a particle in a strongly dispersive medium is approximately proportional to the group index n_{bg}, and can therefore become very large in a slow-light medium.

Acknowledgements

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Appendix A. Consistency of Eqs. (5) and (21)

Since the term involving $\alpha\prime$ in Eq. (21) is essential to our analysis, and in particular to the derivation of Nelson's dispersive contribution (Eq. (25)) to the momentum density, we review here the fact that the expression (5) for the total energy density may be regarded as a consequence of Eq. (21) and Poynting's theorem. We begin by writing Poynting's theorem in its integral form:

$$\begin{aligned}\oint \mathbf{S}\cdot\hat{n}da &= -\int\left[\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mu_0\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t}\right]dV \\ &= -\int\left[\frac{1}{2}\frac{\partial}{\partial t}(\epsilon_0\mathbf{E}^2 + \mu_0\mathbf{H}^2) + \mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}\right]dV \\ &= -\int\dot{u}dV.\end{aligned} \tag{A.1}$$

The integral of the normal component of $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ on the left-hand side is, as usual, over a surface enclosing a volume V, and

$$\dot{u} = \frac{1}{2}\frac{\partial}{\partial t}\left[\epsilon_0\mathbf{E}^2 + \mu_0\mathbf{H}^2\right] + \mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}, \tag{A.2}$$

where u is the density of total energy, that in the field plus that in the medium. Using $\mathbf{P} = N\mathbf{d}$, together with

$$\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}=\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{P})-\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{P}, \tag{A.3}$$

Eq. (21), and $\mathbf{E}=\mathscr{E}_0\exp(-i\omega t)$, we obtain

$$\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}=\frac{1}{4}N(\alpha+\omega\alpha')\frac{\partial}{\partial t}|\mathscr{E}_0|^2 \tag{A.4}$$

when we take the cycle average and use the assumption made in obtaining (21) that $\dot{\mathscr{E}}_0$ is negligible compared to $\omega\mathscr{E}_0$. Then, from $\epsilon(\omega)=\epsilon_0+N\alpha(\omega)$, it follows that

$$\frac{1}{2}\frac{\partial}{\partial t}(\epsilon_0\mathbf{E}^2)+\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t}=\frac{1}{4}\left[\epsilon+\omega\frac{d\epsilon}{d\omega}\right]\frac{\partial}{\partial t}|\mathscr{E}_0|^2, \tag{A.5}$$

from which (5) follows.

A.1. Comments added after original submission

A dispersive contribution to the momentum such as appears in Eq. (25), for example, appears also in earlier work by H. Washimi (see H. Washimi and V.I. Karpman, Sov. Phys. JETP 44 (1976) 528 (1976) and references therein).

When the force (61) is applied to the case of a guest two-level atom in a host medium it reduces to the expression given in Eq. (3) of S.E. Harris, Phys. Rev. Lett. 85 (2000) 4032 if it is assumed that (i) the plane-wave field acting on the atom propagates at the group velocity of the host medium; (ii) the field frequency is sufficiently different from the atom's transition frequency that the term proportional to $\alpha_{R'}$ is negligible; (iii) the medium is sufficiently dispersive that $n_{bg}\gg n_b$; and (iv) absorption is negligible, so that the term proportional to α_I may be ignored.

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[15] Aside from a derivation of some results used in Section 6, the material in this section, especially in connection with the Rayleigh attenuation coefficient, digresses somewhat from the main thrust of the paper. But because Krzysztof Wódkiewicz was very much interested in historical matters of this sort, we feel it is not entirely inappropriate to include it.
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