Laser Physics, Vol. 15, No. 10, 2005, pp. 1432–1438.

Recoil and Photon Momentum in a Dielectric

P. W. Milonni[1,*] and R. W. Boyd[2]

[1] Theoretical Division (T-DOT), Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA
[2] Institute of Optics, University of Rochester, Rochester, New York 14627, USA
*e-mail: pwm@lanl.gov
Received February 15, 2005

Abstract—We consider the recoil momentum of a spontaneously emitting atom in a dispersive dielectric, first from a microscopic approach, in which the dielectric is treated as a collection of atoms and the field is quantized *in vacuum*, and, then, from the macroscopic approach of quantizing the field in the dielectric. The atom's recoil momentum differs from both the Abraham and Minkowski forms of the field momentum, and this difference is explained in terms of a dispersive contribution to the momentum acquired by the medium.

1. INTRODUCTION

The problem of defining the electromagnetic momentum density has been considered by many authors [1]. Discussions typically center on comparisons of the Abraham and Minkowski forms, defined respectively by

$$\mathbf{P}_A = \mathbf{E}\times\mathbf{H}/c^2 \quad (1)$$

and

$$\mathbf{P}_M = \mathbf{D}\times\mathbf{B}. \quad (2)$$

Garrison and Chiao [2] have recently discussed the implications of these forms for the photon momentum in a dispersive dielectric medium and, by quantizing the field in the dielectric, have identified three photon momenta with magnitudes

$$p_c = n(\omega)\hbar\omega/c, \quad (3)$$

$$p_A = v_g(\omega)\hbar\omega/c^2, \quad (4)$$

and

$$p_M = v_g(\omega)n^2(\omega)\hbar\omega/c^2 \quad (5)$$

at frequency ω, where $n(\omega)$ and $v_g(\omega)$ are the refractive index and the group velocity, respectively. The first is the canonical photon momentum, i.e., that associated with the generator of space translations, while the second and third are the Abraham and Minkowski forms, respectively, that follow from the corresponding classical expressions when the field in the dielectric is quantized (see, e.g., Eq. (42) below). Garrison and Chiao discuss the fact that the results of the Jones–Leslie experiment [3], in which the radiation pressure on a mirror in a dielectric is measured, are consistent with the assignment of the canonical momentum to the photons. They find that the reported data deviate from the Abraham and Minkowksi forms by 405 and 22 standard deviations, respectively.

As discussed by Garrison and Chiao, the canonical photon momentum finds physical justification in treatments of the Cerenkov and Doppler effects and in the phase-matching (momentum conservation) conditions of nonlinear optics. Consider, for example, the Doppler shift in the spontaneous radiation from an atom of mass m and transition frequency ω_0 with initial velocity $\mathbf{v}$ and final velocity $\mathbf{v}'$ after emission of a photon of frequency ω. Conservation of energy and linear momentum require, in the nonrelativistic approximation [4],

$$\frac{1}{2}mv'^2 + \hbar\omega = \frac{1}{2}mv^2 + \hbar\omega_0, \quad (6)$$

$$m\mathbf{v}' = m\mathbf{v} - \hbar\mathbf{k}, \quad (7)$$

and, if we take $\mathbf{k} = [n(\omega)\omega/c]\mathbf{s}$, where $\mathbf{s}$ points in the direction of the emitted photon, we obtain the correct (up to terms of order $1/c$) Doppler shift formula

$$\omega = \omega_0\left[1 + \frac{n(\omega)}{c}v\cos\theta\right], \quad (8)$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{s}$ and $-\hbar\mathbf{k}$ is the recoil momentum of the atom.

It is useful for our purposes to summarize a few pertinent remarks by Ginzburg [5]. From Maxwell's equations for a nonmagnetic dielectric, one obtains the well-known relation [5, 6]

$$\mathbf{B}\times(\nabla\times\mathbf{H}) + \mathbf{D}\times(\nabla+\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{D}) + \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) = -[\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}] \quad (9)$$

in the usual notation. The right-hand side is the rate of change of the "mechanical momentum." The last term on the left is the time derivative of the Minkowski field momentum density; however, it is the Abraham form $\mathbf{E}\times\mathbf{H}/c^2$ that is generally regarded as the correct field momentum density [6]. For a nondispersive medium with dielectric constant ϵ, we can identify the Abraham force density $\mathbf{f}^A$ by writing

$$\frac{\partial}{\partial t}\mathbf{D}\times\mathbf{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}\times\mathbf{H}+\mathbf{f}^A, \qquad (10)$$

i.e., $\mathbf{f}^A = (n^2-1)(\partial/\partial t)\mathbf{E}\times\mathbf{H}/c^2$. $\mathbf{f}^A$ can be interpreted as part of a force density $[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}+\mathbf{f}^A]$ acting on the medium. Ginzburg remarks that "It is in reality impossible to question this [Abraham] force notwithstanding that it has as yet not been reliably measured directly. In that way the problem [of choosing between the Minkowski and Abraham forms] would be solved "in favor" of the Abraham tensor." For a nondispersive medium, Abraham and Minkowski photon momenta (4) and (5) become $p_A = \hbar\omega/nc$ and $p_M = n\hbar\omega/c$, while the momentum p^A associated with the Abraham force is $(n^2-1)\hbar\omega/nc$ [5]. Thus,

$$p_M = p_A + p^A = p_c. \qquad (11)$$

From this point of view, the simple derivation of the (correct) Doppler shift formula (8) requires a reinterpretation: $\hbar\mathbf{k}$ is not the momentum ($p_M = p_c$) of the emitted photon but rather the total momentum transferred from the atom to the emitted photon *and* to the medium. In the case of a dispersive medium, however, $p_M \neq p_c$, as noted by Garrison and Chiao. One purpose of this paper is to explain this difference between p_M and the momentum p_c delivered by a photon to an object in the medium, or, in the example we consider, the momentum associated with recoil in spontaneous emission. In particular, we attempt to explain why the Minkowski momentum only gives the correct recoil of an atom—or, more generally, the force on an object in the dielectric—in the case in which dispersion is ignored.

The large literature concerned with the definition of electromagnetic field momentum has dealt mainly with classical fields and nondispersive media. The more recent literature includes detailed classical and quantized-field analyses for dispersive and absorbing dielectrics by Loudon *et al.* [7], which have clarified considerably the difference between field momentum and the momentum actually transferred from an incoming field to an object in the medium. In particular, Loudon advocates a direct calculation of the Lorentz force to determine the forces in and on dielectrics.

Given the fundamental importance of the subject, it is an interesting academic exercise to obtain the recoil momentum of a spontaneously emitting atom in a dielectric without explicit quantization of the electromagnetic field in the dielectric, i.e., by quantizing the field in *free space* and deducing the recoil momentum as a consequence of the fact that, with each atom of the dielectric, a field *propagating at the vacuum speed of light c* is associated. Such an exercise is carried out in the following section, and, in Section 3, we outline the straightforward calculation of the same result by quantizing the macroscopic field in the dielectric. In Section 4 we consider the momentum transferred to the bulk dielectric and discuss the relation of our approach to that of Loudon *et al.* Our conclusions are summarized in Section 5.

2. RECOIL OF AN EMITTER IN A DISPERSIVE DIELECTRIC

We consider a model used recently in a microscopic theory of spontaneous emission in a dispersive dielectric [8]. Here, however, the source atom is taken to be a two-level atom (TLA) [9] with mass m, transition frequency ω_0, and transition dipole moment $\mathbf{d}$, and the dielectric consists of N identical TLAs per unit volume, each having a transition frequency ω' and, for simplicity here, the same transition dipole moment $\mathbf{d}$ as the source atom. The Hamiltonian in the dipole approximation is

$$\hat{H} = \hat{\mathbf{P}}^2/2m + \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \frac{1}{2}\hbar\omega'\sum_l\hat{\sigma}_{zl} + \sum_{\mathbf{k}\lambda}\hbar\omega_k\hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} - \mathbf{d}\hat{\sigma}_x\cdot\hat{\mathbf{E}}(\hat{\mathbf{R}}) - \sum_l\mathbf{d}\hat{\sigma}_{xl}\cdot\hat{\mathbf{E}}(\hat{\mathbf{R}}_l), \qquad (12)$$

$$\hat{\mathbf{E}}(\mathbf{r}) = i\sum_{\mathbf{k}\lambda}\left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2}[\hat{a}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^\dagger_{\mathbf{k}\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}}]\mathbf{e}_{\mathbf{k}\lambda}, \qquad (13)$$

where operators are indicated by carets ($\wedge$). We are using the standard notation involving the Pauli two-state operators ($\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ for the source atom and $\hat{\sigma}_{xl}$, $\hat{\sigma}_{yl}$, $\hat{\sigma}_{zl}$ for the lth dielectric atom) and the photon annihilation ($\hat{a}_{\mathbf{k}\lambda}$) and creation ($\hat{a}^\dagger_{\mathbf{k}\lambda}$) operators. V is the quantization volume, and $\mathbf{e}_{\mathbf{k}\lambda}$ ($\lambda = 1, 2$) is a polarization unit vector for the mode with wave vector $\mathbf{k}$ and polarization λ. $\mathbf{e}_{\mathbf{k}\lambda}$ and $\mathbf{d}$ are taken to be real, and $k = |\mathbf{k}| = \omega_k/c$; i.e., *the field is quantized in free space.* Effects of the dielectric on the rate of emission and the recoil of the source atom are due explicitly to the atoms constituting the dielectric, not to "dressed" photons defined by quantizing the field in the dielectric. $\hat{\mathbf{P}}$ is the linear momentum operator for the center-of-mass motion of the source atom. We ignore, to begin with, any recoil of the dielectric atoms, which in effect are taken to be infinitely massive.

From the Hamiltonian (12), we obtain the Heisenberg equation of motion for the linear momentum $\hat{\mathbf{P}}$ of the source atom:

$$\frac{d\hat{\mathbf{P}}}{dt} = \nabla[\hat{\sigma}_x\mathbf{d}\cdot\hat{\mathbf{E}}(\mathbf{R})]. \qquad (14)$$

In the dipole approximation, the spatial variations of the field are small over the dimensions of the atom, so that $\mathbf{R}$ in Eq. (14) is regarded in effect as the expectation value of the operator $\hat{\mathbf{R}}$. We write $\hat{\sigma}_x = \hat{\sigma} + \hat{\sigma}^\dagger$, where $\hat{\sigma}$ and $\hat{\sigma}^\dagger$ are, respectively, the lowering and raising operators for the two-level source atom and $\hat{\mathbf{E}}(\mathbf{R}) = \hat{\mathbf{E}}^{(+)}(\mathbf{R}) + \hat{\mathbf{E}}^{(-)}(\mathbf{R})$, where $\hat{\mathbf{E}}^{(+)}(\mathbf{R})$ and $\hat{\mathbf{E}}^{(-)}(\mathbf{R})$ are, respectively, the photon annihilation and creation parts of the field. $\hat{\mathbf{E}}^{(+)}(\mathbf{R})$ and $\hat{\mathbf{E}}^{(-)}(\mathbf{R})$ are given by

$$\hat{\mathbf{E}}^{(\pm)}(\mathbf{R}) = \hat{\mathbf{E}}_0^{(\pm)}(\mathbf{R}) + \hat{\mathbf{E}}_d^{(\pm)}(\mathbf{R}), \tag{15}$$

where $\hat{\mathbf{E}}_0(\mathbf{R})$ is the solution of the homogeneous (source-free) wave equation for the field and $\hat{\mathbf{E}}_d(\mathbf{R})$ is the part of the field due to all the dipole sources. Writing (14) in normal order for the field operators and taking the expectation value in an initial state in which there are no photons in the field, we have

$$\frac{d}{dt}\langle\hat{\mathbf{P}}\rangle = 2\Re\nabla\langle\mathbf{d}\hat{\sigma}^\dagger\cdot\hat{\mathbf{E}}_d^{(+)}(\mathbf{R})\rangle. \tag{16}$$

Using the formal solution of the Heisenberg equation for $\hat{a}_{\mathbf{k}\lambda}$ in Eq. (13) for the electric field operator, we obtain, of course, formally the same expression for the field as in classical electrodynamics for a collection of dipoles [10]:

$$\hat{\mathbf{E}}_d(\mathbf{r}, t) = \hat{\mathbf{E}}_s(\mathbf{r}, t) + \sum_l \nabla\times\nabla\times\frac{\mathbf{d}\hat{\sigma}_{xl}(t-(|\mathbf{r}-\mathbf{r}_l|/c)}{|\mathbf{r}-\mathbf{r}_l|}, \tag{17}$$

or, in the rotating-wave approximation (RWA) [9] of replacing $\hat{\sigma}_{xl}$ by $\hat{\sigma}_l$ in $\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}, t)$,

$$\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}, t) \cong \hat{\mathbf{E}}_s^{(+)}(\mathbf{r}, t) + \sum_l \nabla\times\nabla\times\frac{\mathbf{d}\hat{\sigma}_l(t-|\mathbf{r}-\mathbf{r}_l|/c)}{|\mathbf{r}-\mathbf{r}_l|}, \tag{18}$$

where $\hat{\mathbf{E}}_s(\mathbf{r}, t)$ is the field from the source atom and the second term on the right-hand side of (17) is the sum of the fields of the dielectric atoms.

The source atom is singled out in (17) and (18) because it is initially excited. The other atoms constituting the dielectric are initially unexcited, each of them has a dipole moment induced by the field acting on it, and each frequency component of this dipole moment is linearly proportional to the field at the same frequency. In the RWA, at each field frequency $\hat{\sigma}_l(t)$ is proportional to the photon annihilation part of the field at $\mathbf{r}_l$. But because of normal ordering, only the source part of this field, $\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}_l, t)$, contributes to the expectation value in (16), so that, in effect, we can ignore the source-free (vacuum) field and write the equation

$$\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}, \omega) = \hat{\mathbf{E}}_s^{(+)}(\mathbf{r}, \omega) + \alpha(\omega)\sum_l \nabla\times\nabla\times\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}_l, \omega)\frac{e^{i\omega|\mathbf{r}-\mathbf{r}_l|/c}}{|\mathbf{r}-\mathbf{r}_l|} \tag{19}$$

for each frequency component ω of the field, where $\alpha(\omega)$ is the polarizability at frequency ω. In the continuum approximation, we replace this equation by

$$\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}, \omega) = \hat{\mathbf{E}}_s^{(+)}(\mathbf{r}, \omega) + N\alpha(\omega)\int_a^\Sigma d^3r'\nabla\times\nabla\times\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}', \omega)\frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{20}$$

where $k_0 = \omega/c$. The integration is over the volume bounded by the surface Σ of the dielectric, and the subscript a on the integral sign indicates that a small sphere of radius a about $\mathbf{r}$ is excluded from the integration; $a \longrightarrow 0$ after the integration.

Equation (20) is just a statement of the superposition principle: the total field at any point in the medium is the field from the source atom plus the field from the atoms making up the dielectric in which the source atom is placed. It was derived by quantizing the field *in vacuum*; the field of each atom propagates with the vacuum speed of light c. This equation has exactly the same form as the *classical* integral equation that is the starting point in the proof of the Ewald–Oseen extinction theorem given in *Born and Wolf* [11], except that, in the latter, the first term on the right-hand side is an externally applied field incident on the medium, whereas in Eq. (20) it is the field from an atom *inside* the medium. If we assume a solution of Eq. (20) of the form

$$\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}, \omega) = \hat{\mathbf{A}}e^{i\mathbf{K}\cdot\mathbf{r}}, \tag{21}$$

with $\mathbf{K}\cdot\hat{\mathbf{A}} = 0$, then, as shown in *Born and Wolf*, the integral on the right-hand side of (20) is equal to $[4\pi N\alpha(\omega)][K^2 + 2k_0^2]/[K^2 - k_0^2]$ plus a surface integral that must, for consistency, cancel the first term on the left-hand side. Therefore,

$$\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}, \omega) = \frac{4\pi N\alpha(\omega)}{3}\frac{K^2+2k_0^2}{K^2-k_0^2}\hat{\mathbf{E}}_d^{(+)}(\mathbf{r}, \omega) \tag{22}$$

or

$$K^2(\omega) = n^2(\omega)k_0^2 = n^2(\omega)\frac{\omega^2}{c^2}, \tag{23}$$

where $4\pi N\alpha(\omega)/3 = [n^2(\omega) - 1]/[n^2(\omega) + 1]$, i.e., $n(\omega)$ is the refractive index. In other words, at each frequency ω, the superposition of the fields from all the atoms of the medium, each of which propagates at the velocity c, results in the phase velocity $c/n(\omega)$ for the total field. This is all that will be required to infer that a photon in the medium has a linear momentum of magnitude $n(\omega)\hbar\omega/c$. In particular, it is not necessary for this purpose to quantize the field in the dielectric rather than in vacuum.

Let us return now to the calculation of the recoil momentum of the source atom. The formal solution of the Heisenberg equation of motion for $\hat{a}_{\mathbf{k}\lambda}(t)$ gives

$$\hat{\mathbf{E}}_d^{(+)}(\mathbf{R}, t) = \frac{i}{4\pi^2}\int d^3k_0\omega[\mathbf{d} - (\mathbf{d}\cdot\mathbf{k}_0)\mathbf{k}_0/k_0^2] \times \int_0^t dt'\left[\hat{\sigma}(t')e^{-i\mathbf{k}_0\cdot\mathbf{R}} + \sum_l \hat{\sigma}_l(t')e^{-i\mathbf{k}_0\cdot\mathbf{R}_l}\right]e^{i\omega_k(t'-t)}, \tag{24}$$

which, in the dielectric continuum limit, is just another way of writing (20). The considerations leading from (20) to (21) and (23) do not give us an explicit solution for $\hat{\mathbf{E}}_d^{(+)}(\mathbf{R}, t)$. But they determine how each frequency component of $\hat{\mathbf{E}}_d^{(+)}(\mathbf{R}, t)$ varies with $\mathbf{R}$ and, for the purpose of calculating $\langle d\hat{\mathbf{P}}/dt\rangle$ as described below, allow us to write (16) as

$$\frac{d}{dt}\langle\hat{\mathbf{P}}\rangle = -\frac{d^2}{2\pi^2}\Re\int d^3k_0\omega n(\omega)\mathbf{k}_0[1-(\mathbf{d}\cdot\mathbf{k}_0)^2/d^2k_0^2] \times \int_0^t dt'[\langle\hat{\sigma}^\dagger(t)\hat{\sigma}(t')\rangle + N\int d^3R'\langle\hat{\sigma}^\dagger(t)\hat{\sigma}(\mathbf{R}', t')\rangle e^{-i\mathbf{k}_0\cdot(\mathbf{R}'-\mathbf{R})}]e^{i\omega(t'-t)}. \tag{25}$$

The Heisenberg equation of motion for $\hat{\sigma}(\mathbf{R}', t)$ is

$$\dot{\hat{\sigma}}(\mathbf{R}', t) = -i\omega'\hat{\sigma}(\mathbf{R}', t) - \frac{i}{\hbar}\hat{\sigma}_z(\mathbf{R}', t)\mathbf{d}\cdot\hat{\mathbf{E}}^{(+)}(\mathbf{R}', t). \tag{26}$$

The second term on the right has contributions from the field of the dielectric at $\mathbf{R}'$ plus the fields of all the other atoms. The contribution from the atom at $\mathbf{R}'$, in the Markovian or Weisskopf–Wigner approximation, is responsible for radiative damping of the dielectric atom at $\mathbf{R}'$; i.e., it contributes $-\beta'\hat{\sigma}(\mathbf{R}', t)$, where $\beta' = d^2\omega'^3/6\pi\epsilon_0\hbar c^3$. We ignore radiative shifts here, which for our purposes can be assumed to be included in the definitions of the transition frequencies. The effect of all the other atoms is complicated, and we approximate it by including only the effect of the source atom. For a dilute dielectric, the dielectric atoms make a relatively small contribution because they are unexcited; for this reason we also approximate $\hat{\sigma}_z(\mathbf{R}', t)$ by -1, assuming that, with high probability, every dielectric atom remains forever unexcited. Thus,

$$\dot{\hat{\sigma}}(\mathbf{R}', t) \cong -i(\omega' - i\beta')\hat{\sigma}(\mathbf{R}', t) + \frac{i}{\hbar}\mathbf{d}\cdot\hat{\mathbf{E}}_s^{(+)}(\mathbf{R}', t). \tag{27}$$

In the RWA, and with $\dot{\hat{\sigma}}(t) \cong -i\omega_0\hat{\sigma}(t)$ and $\ddot{\hat{\sigma}}(t) \cong -\omega_0^2\hat{\sigma}(t)$, we have

$$\frac{1}{\hbar}\mathbf{d}\cdot\hat{\mathbf{E}}_s^{(+)}(\mathbf{R}', t) \cong \frac{d^2}{\hbar}k_0^3F(\mathbf{R}')\hat{\sigma}(t), \tag{28}$$

where we now take $\mathbf{R} = 0$ for the coordinate of the source atom and define

$$F(\mathbf{R}) = \left(\frac{1}{k_0R}\left[1 - \frac{(\mathbf{d}\cdot\mathbf{R})^2}{d^2R^2}\right] + \left[\frac{i}{k_0^2R^2} - \frac{1}{k_0^3R^3}\right]\left[1 - 3\frac{(\mathbf{d}\cdot\mathbf{R})^2}{d^2R^2}\right]\right)e^{ik_0R}. \tag{29}$$

Then,

$$\dot{\hat{\sigma}}(\mathbf{R}', t) \cong -i(\omega' - i\beta')\hat{\sigma}(\mathbf{R}', t) + i\frac{d^2}{\hbar}k_0^3F(\mathbf{R}')\hat{\sigma}(t) \tag{30}$$

and

$$\hat{\sigma}(\mathbf{R}', t) \cong \frac{d^2}{\hbar}\frac{k_0^3F(\mathbf{R}')}{\omega' - \omega_0}\hat{\sigma}(t), \tag{31}$$

assuming that $|\omega' - \omega_0|$ is large enough for this adiabatic approximation and for β' to be neglected. In these approximations, (25) is replaced by

$$\frac{d}{dt}\langle\hat{\mathbf{P}}\rangle = -\frac{d^2}{2\pi^2}\Re\int d^3k_0\omega n(\omega)\mathbf{k}_0\left[1 - \frac{(\mathbf{d}\cdot\mathbf{k}_0)^2}{d^2k_0^2}\right] \times \left[1 + \frac{Nd^2/\hbar}{\omega' - \omega_0}k_0^3\int d^3RF(\mathbf{R})e^{-i\mathbf{k}_0\cdot\mathbf{R}}\right] \times \int_0^t dt'\langle\hat{\sigma}^\dagger(t)\hat{\sigma}(t')\rangle e^{i\omega(t'-t)}. \tag{32}$$

The integral over time is evaluated in the standard Markovian or Weisskopf–Wigner approximation for $t \gg \omega_0^{-1}$:

$$\int_0^t dt'\langle\hat{\sigma}^\dagger(t)\hat{\sigma}(t')\rangle e^{i\omega(t'-t)} \cong \langle\hat{\sigma}^\dagger(t)\hat{\sigma}(t)\rangle\int_0^t dt' e^{i(\omega-\omega_0)(t'-t)} \longrightarrow \pi p_+(t)\delta(\omega - \omega_0), \tag{33}$$

where we drop a Cauchy principal part term that gives a "Lamb shift," which we are ignoring, and $p_+(t)$ is the probability at time t that the source TLA is in the upper state. For our initially excited source atom, $p_+(t) = \exp(-A't)$, where A' is the spontaneous emission rate *in the dielectric.* Then, for $t \gg 1/A'$,

$$\begin{aligned}\langle\hat{\mathbf{P}}\rangle &= -\frac{d^2}{2\pi A'}\Re\int d^3k\omega n(\omega)\mathbf{k}\left[1-\frac{(\mathbf{d}\cdot\mathbf{k})^2}{d^2k^2}\right]\\ &\times\left[1+\frac{Nd^2/\hbar}{\omega'-\omega_0}k_0^3\int d^3RF(\mathbf{R})e^{-i\mathbf{k}\cdot\mathbf{R}}\right]\delta(\omega-\omega_0)\\ &= -\frac{d^2\omega_0^3}{2\pi A'\hbar c^3}\int d\Omega_{\mathbf{k}_0}[n(\omega_0)\hbar\mathbf{k}_0]\left[1-\frac{(\mathbf{d}\cdot\mathbf{k}_0)^2}{d^2k_0^2}\right]\\ &\times\left[1+\frac{Nd^2/\hbar}{\omega'-\omega_0}k_0^3\int d^3RF(\mathbf{R})e^{-i\mathbf{k}_0\cdot\mathbf{R}}\right],\end{aligned} \tag{34}$$

where $\int d\Omega_{\mathbf{k}_0}[\ldots]$ is the integral over all solid angles about $\mathbf{k}_0$ and now $k_0 = \omega_0/c$. Obviously the integral over solid angles vanishes, as it must because the direction of photon emission is equally likely in directions $\mathbf{k}_0$ and $-\mathbf{k}_0$. However, it is straightforward to show, using the same approximations as above, that

$$\begin{aligned}\langle\hat{\mathbf{P}}^2\rangle &= \frac{d^2\omega_0^3}{2\pi A'\hbar c^3}n^2(\omega_0)\hbar^2k_0^2\int d\Omega_{\mathbf{k}_0}\left[1-\frac{(\mathbf{d}\cdot\mathbf{k}_0)^2}{d^2k_0^2}\right]\\ &\times\left[1+\frac{Nd^2/\hbar}{\omega'-\omega_0}\Re k_0^3\int d^3RF(\mathbf{R})e^{-i\mathbf{k}_0\cdot\mathbf{R}}\right].\end{aligned} \tag{35}$$

Now,

$$\int d\Omega_{\mathbf{k}_0}\left[1-\frac{(\mathbf{d}\cdot\mathbf{k}_0)^2}{d^2k_0^2}\right] = \frac{8\pi}{3} \tag{36}$$

and

$$\int d\Omega_{\mathbf{k}_0}\left[1-\frac{(\mathbf{d}\cdot\mathbf{k}_0)^2}{d^2k_0^2}\right]e^{-i\mathbf{k}_0\cdot\mathbf{R}} = 4\pi F_i(\mathbf{R}), \tag{37}$$

where $F_i(\mathbf{R})$ is the imaginary part of $F(\mathbf{R})$. Therefore,

$$\begin{aligned}&\langle\hat{\mathbf{P}}^2\rangle\\ &= \frac{4d^2\omega_0^3}{3A'\hbar c^3}n^2(\omega_0)\hbar^2k_0^2\left[1+\frac{2\pi Nd^2/\hbar}{\omega'-\omega_0}k_0^3\Im\int d^3RF^2(\mathbf{R})\right]\\ &= \frac{A}{A'}\left[n(\omega_0)\frac{\omega_0}{c}\right]^2[1+\delta nk_0^3\Im\int d^3RF^2(\mathbf{R})],\end{aligned} \tag{38}$$

where A is the spontaneous emission rate of the source atom in free space and $\delta n = 2\pi Nd^2/[\hbar(\omega'-\omega_0)]$ is the refractive index at frequency ω_0 for a dilute dielectric consisting of N TLAs per unit volume, each having transition dipole moment $\mathbf{d}$ and transition frequency ω'.

We have assumed that all the transition moments point in the same direction; whereas, to model an isotropic dielectric, we should allow the dielectric atoms to have transition moments pointing in all three directions. Doing this, and averaging over the relative orientations of the source- and dielectric-atom dipole moments, or, alternatively, carrying out the calculation as above but allowing the dielectric atoms to make $\Delta m = 0, \pm1$ transitions, for instance, we obtain [8]

$$A[1+\delta nk_0^3\Im\int d^3RF^2(\mathbf{R})] = \left[1+\frac{7}{3}\delta n\right]A, \tag{39}$$

which, as shown in [8], is the spontaneous emission rate A of the source atom in the (dilute) dielectric; i.e., it is the small-δn approximation to $nA[(n^2+2)/3]^2$, the spontaneous emission rate, including the Lorentz–Lorenz local field correction, in a dielectric. Thus,

$$\langle\hat{\mathbf{P}}^2\rangle = n^2(\omega_0)\frac{\hbar^2\omega_0^2}{c^2}. \tag{40}$$

The recoil momentum of the source atom has the expected magnitude $p_c = n(\omega_0)\hbar\omega_0/c$. Note that, although our derivation includes a local field correction at the source atom, this correction has no effect on the source atom's recoil momentum.

3. RECOIL CALCULATION BY FIELD QUANTIZATION IN THE DIELECTRIC

We now calculate the recoil momentum using the single-atom Hamiltonian

$$\begin{aligned}\hat{H} &= \hat{\mathbf{P}}^2/2m+\frac{1}{2}\hbar\omega_0\hat{\sigma}_z\\ &+\sum_{\mathbf{k}\lambda}\hbar\omega_k\hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda}-\mathbf{d}\hat{\sigma}_x\cdot\hat{\mathbf{E}}(\hat{\mathbf{R}})\end{aligned} \tag{41}$$

with the electric field quantized not in free space but in the dielectric [12]:

$$\hat{\mathbf{E}}(\mathbf{r},t) = i\sum_{\mathbf{k}\lambda}\left(\frac{\hbar\omega}{2nn_g\epsilon_0V}\right)^{1/2}\hat{a}_{\mathbf{k}\lambda}(t)\mathbf{e}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}+\text{h.c.}, \tag{42}$$

where $k = n(\omega)\omega/c$, $n_g = n(\omega)+\omega(dn/d\omega)$, and, of course, the annihilation and creation operators now refer to the "dressed" photons of the dielectric. We assume that the field frequencies that will contribute to recoil are far removed from any absorption resonances, so that the field is quantized in a lossless (but dispersive) dielectric. Since the role of normal ordering and other aspects of the calculation here are straightforward and much the same as in the preceding section, we proceed immediately to the RWA expression for the rate of change of the expectation value of $\hat{\mathbf{P}}^2$:

$$\frac{d}{dt}\langle\hat{\mathbf{P}}^2\rangle = 2\Re\hbar\sum_{\mathbf{k}\lambda}\left(\frac{\hbar\omega}{2nn_g\epsilon_0 V}\right)k^2(\mathbf{d}\cdot\mathbf{e}_{\mathbf{k}\lambda})^2 \times\int_0^t dt'\langle\hat{\sigma}^\dagger(t)\hat{\sigma}(t')\rangle e^{i\omega(t'-t)}. \quad (43)$$

Going to the mode continuum limit and using $dk = (n_g/c)d\omega$ and (33), we have

$$\frac{d}{dt}\langle\hat{\mathbf{P}}^2\rangle = n(\omega_0)A\left(\frac{n(\omega_0)\hbar\omega_0}{c}\right)^2 p_+(t) = n(\omega_0)A\left(\frac{n(\omega_0)\hbar\omega_0}{c}\right)^2 e^{-A't}, \quad (44)$$

where again $A = d^2\omega_0^3/3\pi\epsilon_0 c^3$ is the free-space spontaneous emission rate. Without any local field correction, $A' = n(\omega_0)A$, and (40) follows. If there is a local field correction $\mathcal{L}$ such that $A' = \mathcal{L}nA$, it should be included in the field, and, therefore, (44) should be multiplied by $\mathcal{L}$. In other words, the recoil momentum is $n\hbar\omega_0/c$ regardless of whether there is a local field correction, just as in the calculation in the preceding section.

4. MOMENTUM TRANSFER TO THE DIELECTRIC

Let us consider now the force on the dielectric atoms. Since they are assumed to be unexcited, this force must arise from the field of the source atom or, more generally, any applied field. The ith component of the force on the bulk dielectric is, in the continuum approximation and employing the summation convention for repeated indices,

$$F_i = \frac{d\mathcal{P}_i}{dt} = \int P_j\frac{\partial E_j}{\partial x_i}d^3r, \quad (45)$$

where $\mathcal{P}_i$ is the momentum imparted to the dielectric and P_j is the jth component of the macroscopic polarization in the dielectric. For simplicity we will formulate the discussion here classically to begin with.

Since the field from the source atom is time-dependent, we write

$$E_j(\mathbf{r},t) = \mathcal{E}_j(\mathbf{r},t)e^{-i\omega_0 t} = e^{-i\omega_0 t}\int_{-\infty}^{\infty}d\Delta\tilde{\mathcal{E}}_j(\mathbf{r},\Delta)e^{-i\Delta t} \quad (46)$$

and

$$P_j(\mathbf{r},t) = \epsilon_0\int_{-\infty}^{\infty}d\Delta\chi(\omega_0+\Delta)\tilde{\mathcal{E}}(\mathbf{r},\Delta)e^{-i(\omega_0+\Delta)t}, \quad (47)$$

where $\chi(\omega)$ is the susceptibility, which is taken to be independent of $\mathbf{r}$ (no spatial dispersion). Assuming that $\mathcal{E}_j(\mathbf{r},t)$ is slowly varying compared with $\exp(-i\omega_0 t)$, we have

$$P_j(\mathbf{r},t) = \epsilon_0\int_{-\infty}^{\infty}d\Delta\chi(\omega_0) + \Delta\chi'(\omega_0)+\ldots\tilde{\mathcal{E}}(\mathbf{r},\Delta)e^{-i(\omega_0+\Delta)t} \cong \epsilon_0\chi E_j + i\epsilon_0\chi'\frac{\partial\mathcal{E}_j}{\partial t}e^{-i\omega_0 t}, \quad (48)$$

where χ and $\chi' = d\chi/d\omega$ are evaluated at $\omega = \omega_0$. Then,

$$\mathcal{P}_j\frac{\partial E_j}{\partial x_i} = \frac{1}{2}\epsilon_0\chi\frac{\partial}{\partial x_i}\mathbf{E}^2 + \frac{i}{4}\epsilon_0\chi'\left(\frac{\partial\mathcal{E}_j}{\partial t}\frac{\partial\mathcal{E}_j^*}{\partial x_i} - \frac{\partial\mathcal{E}_j^*}{\partial t}\frac{\partial\mathcal{E}_j}{\partial x_i}\right) \quad (49)$$

upon cycle averaging.

In the case of a plane wave,

$$E_j(\mathbf{r},t) = \mathcal{E}_{j0}(t)e^{-i\omega_0 t}e^{i\mathbf{k}_0\cdot\mathbf{r}}, \quad (50)$$

(49) becomes

$$\mathcal{P}_j\frac{\partial\mathcal{E}_j}{\partial x_i} = \frac{1}{4}\epsilon_0\chi' k_{0i}\left(\frac{\partial\mathcal{E}_{j0}}{\partial t}\mathcal{E}_{j0}^* + \frac{\partial\mathcal{E}_{j0}^*}{\partial t}\mathcal{E}_{j0}\right), \quad (51)$$

which implies, from (45) and $\chi = n^2 - 1$,

$$\frac{d\mathbf{P}}{dt} = \frac{1}{4}\epsilon_0\chi'\mathbf{k}_0\frac{\partial}{\partial t}\sum_{j=1}^{3}|\mathcal{E}_j|^2 V = \epsilon_0 n\frac{dn}{d\omega}\left(\frac{n\omega_0}{c}\right)\frac{\mathbf{k}_0}{k_0}\frac{\partial}{\partial t}\mathbf{E}^2 V. \quad (52)$$

Therefore, the momentum density of the dielectric has the magnitude

$$g_{\mathcal{D}} = \epsilon_0\left(\frac{n^2\omega_0}{c}\right)\frac{dn}{d\omega}\mathbf{E}^2. \quad (53)$$

To phrase the discussion in terms of photons, we use Eq. (42) to write, for a single plane-wave mode of the field,

$$\mathbf{E}^2 = \frac{\hbar\omega_0}{nn_g\epsilon_0 V}\hat{a}^\dagger\hat{a} \quad (54)$$

after dropping the term associated with zero-point energy and momentum. Then, (53) implies that each photon in the bulk dielectric imparts to the dielectric a momentum of magnitude

$$p_{\mathcal{D}} = \frac{n}{n_g}\omega\frac{dn}{d\omega}\frac{\hbar\omega}{c}. \quad (55)$$

Now, consider the conservation of momentum when the source atom emits a photon and recoils. If we take the direction of the emitted photon to be positive, the source atom's recoil momentum is $-hn\omega_0/c$, as shown earlier. The momentum of the emitted field is given by the Abraham momentum $\hbar\omega_0/n_g c$, while the medium picks up a momentum $(n^2-1)\hbar\omega_0/n_g c$ corresponding to the Abraham force $(n^2-1)(\partial/\partial t)(\mathbf{E}\times\mathbf{B})/c^2$ [5]. Because of dispersion, the medium also picks up the momentum $p_{\mathcal{D}}$. Thus, the conservation of linear momentum can be expressed in the form

$$\left(-n+\frac{1}{n_g}+\frac{n^2-1}{n_g}+\frac{n}{n_g}\omega\frac{dn}{d\omega}\right)\frac{\hbar\omega_0}{c} = 0, \tag{56}$$

which can be written equivalently as

$$-p_c+p_M+p_{\mathcal{D}} = 0. \tag{57}$$

Note that the momentum densities g_M or $g_{\mathcal{D}}$ can be negative but that their sum is always positive for $n > 0$.

5. DISCUSSION

Equation (56) explains why neither the Abraham nor the Minkowski momenta for a photon give the momentum imparted to an object by a field in a dispersive dielectric medium [2]. Both of these momenta are part of a momentum conservation condition that must also include the dispersive component $p_{\mathcal{D}}$ of the momentum picked up by the medium; the latter, plus the Minkowski momentum, is equal in magnitude to the canonical momentum p_c. The sum $p_M + p_{\mathcal{D}}$ $(=p_c)$ may be regarded in our example as the momentum imparted to the field and the bulk dielectric.

In their treatment of the propagation characteristics of energy and momentum in dispersive (and absorbing) dielectrics, Loudon, Allen, and Nelson have also obtained, in a rather different way, a momentum of form (53), as did Nelson earlier [7]. These treatments, based on a Lagrangian formulation, arrive at such a form as a dispersive contribution to the pseudomomentum, a quantity that is conserved (in the absence of dissipation) when the dielectric is homogeneous. The pseudomomentum combines with the momentum to give the "wave momentum," which, in the terminology of this paper, is the canonical momentum of Garrison and Chiao [2]. In essence this dispersive component of the pseudomomentum is a dispersive contribution to the Minkowksi momentum.

Note added in proof: Campbell *et al.* [Phys. Rev. Lett. **94**, 170403 (2005)] have recently reported the results of an experiment confirming that an atom in a dispersive medium acquires a recoil momentum of magnitude $(\hbar\omega_0/c)n(\omega_0)$ when it absorbs a photon of frequency ω_0, in agreement with the conclusions of this paper.

ACKNOWLEDGMENTS

We thank Professor Eberly for innumerable discussions about topics relating to this work, which we present here partly in the hope of stimulating more discussions in the future. We also thank J.C. Garrison and R.Y. Chiao for correspondence concerning the Abraham and Minkowski momenta and P.R. Berman for insights into radiative processes in dielectrics. One of us (RWB) gratefully acknowledges support by the National Science Foundation and by the DARPA/DSO slow-light program.

REFERENCES

1. I. Brevik, Phys. Rep. **52**, 133 (1979); V. L. Ginzburg, *Theoretical Physics and Astrophysics* (Pergamon, Oxford, 1979), Chap. XII.
2. J. C. Garrison and R. Y. Chiao, Phys. Rev. A **70**, 053 826 (2004).
3. R. V. Jones and B. Leslie, Proc. R. Soc. London A **360**, 347 (1978).
4. E. Fermi, Rev. Mod. Phys. **4**, 87 (1932).
5. V. L. Ginzburg, *Theoretical Physics and Astrophysics* (Pergamon, Oxford, 1979), pp. 281–288.
6. J. D. Jackson, *Classical Electrodynamics* (Wiley, N. Y., 1975), p. 240.
7. R. Loudon, J. Mod. Opt. **49**, 821 (2002); R. Loudon, L. Allen, and D. F. Nelson, Phys. Rev. E **55**, 1071 (1997); D. F. Nelson, Phys. Rev. A **44**, 3985 (1991).
8. P. R. Berman and P. W. Milonni, Phys. Rev. Lett. **92**, 053601 (2004).
9. L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).
10. M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1969; Nauka, Moscow, 1973), p. 104.
11. M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1969; Nauka, Moscow, 1973), p. 105, Eq. (4).
12. K. J. Blow, R. Loudon, S. J. D. Phoenix, and T. J. Sheperd, Phys. Rev. A **42**, 4102 (1990).