Optics Communications 103 (1993) 211–215
North-Holland

OPTICS
COMMUNICATIONS

Noise properties of quantum amplifiers with frequency-dependent gain

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Received 27 May 1993

The power spectrum is calculated of the output of an optical amplifier with frequency-dependent gain for the case in which the input consists of one excited mode that is in an arbitrary quantum state and an infinite set of other modes that are in the vacuum state. The power spectrum is found to be independent of the particular quantum state of the excited input mode and to depend only on the shot-noise level of the amplified output and on the number of noise photons generated by the amplifier at the upper and lower frequency sidemodes of the excited input mode. For the case of a quantum-noise-limited amplifier and a coherent-state input, the results agree with the single-mode treatment of the amplifier only for the case in which the gain experienced by the sidemodes is equal to that experienced by the excited input mode.

1. Introduction

The quantum-mechanical properties of the electromagnetic field set a fundamental limitation to the noise level of optical beams [1]. Theoretical treatments of the fluctuation properties of the electromagnetic field often assume that only the mode at the carrier frequency of the field is excited and focus the attention to the quantum-mechanical ensemble fluctuations of the photon number of this excited mode. On the other hand, fluctuation properties of optical beams are usually studied experimentally by the technique of spectral analysis where the beam is detected and the temporal variations of the resulting photocurrent are resolved in frequency space by a spectrum analyzer. The noise power of the photocurrent measured in some frequency band is often compared to the variance of the quantum-mechanical ensemble fluctuations of the photon number of the excited mode by normalizing both quantities to the respective shot-noise levels. However, the results of a spectral analysis experiment actually depend on the properties of the field at upper and lower frequency sidebands that are displaced by the spectrum-analyzer frequency from the carrier frequency of the field. For example, the standard shot of the noise of a coherent laser beam arises from the beating between the excited mode of the field and the vacuum fluctuations at its frequency sidebands [1]. Hence, single-mode theories are not necessarily adequate for the description of measurements based on spectral analysis, and in general one should include all the modes of the field in the theoretical analysis [2–4]. This is particularly important in cases where the properties of the field are modified by some frequency-dependent interaction such as a resonant interaction in an atomic vapor.

In this paper, we emphasize by explicit calculation the importance of multi-mode treatment of the field for the case of an optical amplifier with frequency-dependent gain. A particular example of such an amplifier is provided by a two-beam-coupling amplifier utilizing an atomic vapor [5], where the gain experienced by a weak probe beam in the presence of a strong pump beam can vary considerably as a function of the frequency of the probe beam. In particular, we calculate the power spectrum corresponding to the intensity fluctuations of the amplified field. A similar treatment could easily be extended to noise

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reduction (squeezing [6] experiments utilizing parametric interactions. Several general theoretical treatments of nonlinear interactions which give rise to squeezing use a multi-mode treatment of the field to calculate the fluctuation properties of the field quadratures [3]. However, more detailed theoretical treatments of these interactions such as four-wave mixing [7] in an atomic vapor [8] typically use a few-mode description of the field. Hence, such treatments are unable to account for any effects due to the frequency-dependence of the interaction. The quantum-mechanical limits on the performance of optical amplifiers have been investigated by several authors beginning with the treatment by Shimoda, Takahasi and Townes [9]. More recent treatments of quantum amplifiers determine the minimum amount of noise added by the process of amplification to the input field by simply requiring that the output field operators satisfy the appropriate commutation relations [2,10,11]. With the exception of the paper by Caves [2], which introduces several important general concepts for the case of multimode fields, theoretical treatments of quantum amplifiers consider the quantum-mechanical ensemble fluctuations of single-mode input and output fields.

2. Theoretical formulation

We consider a schematic setup shown in fig. 1

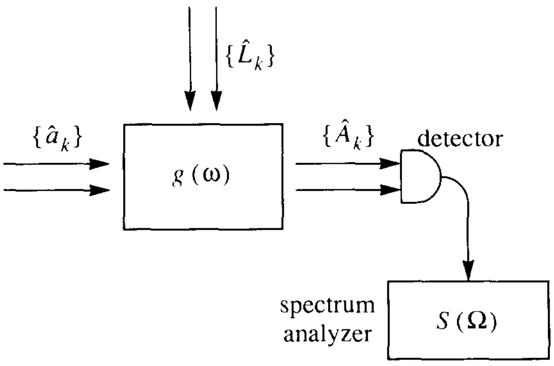

Fig. 1. A quantum amplifier with frequency-dependent gain $g(\omega)$. The amplified field is detected and the resulting photocurrent is fed into a spectrum analyzer to determine the power spectrum $S(\Omega)$ of the fluctuations of the photocurrent.

where a multi-mode input field interacts with an amplifier with frequency-dependent gain $g(\omega)$. The amplified beam is detected and the resulting photocurrent is spectrally analyzed. For simplicity, we assume that the photodetector is ideal, i.e., that it has unit quantum efficiency. More realistic cases of detectors with sub-unity quantum efficiency can be easily treated by coupling the amplified modes to vacuum modes by an appropriate beam splitter [12]. We also assume that all the modes are polarized in the same direction. The annihilation operators for the input and output modes are denoted by $\hat{a}_k$ and $\hat{A}_k$, respectively. The input modes satisfy the commutation relations [13]

$$[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{kk'}, \quad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}^\dagger_k, \hat{a}^\dagger_{k'}] = 0 . \tag{1}$$

In order to preserve analogous commutation relations for the output modes, the annihilation operators for the output modes are expressed in terms of the annihilation operators for the input modes and Langevin operators $\hat{L}_k$, which represent the reservoir degrees of freedom of he amplifier, as [2,10]

$$\hat{A}_k = g(\omega_k)\hat{a}_k + \hat{L}_k . \tag{2}$$

The Langevin operators are assumed to be uncorrelated from the input operators. Note that the detailed form of the Langevin operator depends on the internal properties of the amplifier. Its form is known, for example, for parametric amplifiers [14], Raman amplifiers [15], and two-beam-coupling amplifiers [16]. We consider the special case, which closely approximates several practical situations, where the input to the amplifier is in an arbitrary quantum state for the mode at the carrier frequency of the field, which we denote by $k=0$, and is in the vacuum state for all other input modes. The input state is thus given by $|\Psi_{k=0}, 0_{k\neq 0}\rangle$. To keep the following treatment as general as possible, we do not specify the internal state of the amplifier, i.e., the state onto which the Langevin operators operate is assumed to be arbitrary. However, we assume that the Langevin operators are governed by gaussian processes and hence the moments of the Langevin operator with unequal powers of $\hat{L}_k$ and $\hat{L}^\dagger_k$ vanish.

To calculate the power spectrum of the detected photocurrent, we first calculate the photocurrent correlation function given by [17]

$$C(\tau)=e\langle\hat{I}\rangle\,\delta(\tau)+\langle\hat{I}\rangle^2 g^{(2)}(\tau)\,, \tag{3}$$

where $\langle\hat{I}\rangle$ is the average photocurrent, $\delta(\tau)$ is a delta function, and $g^{(2)}(\tau)$ is a second-order intensity correlation function in normal order given by [18]

$$g^{(2)}(\tau)=\frac{\langle\hat{E}^{(-)}(t)\,\hat{E}^{(-)}(t+\tau)\,\hat{E}^{(+)}(t+\tau)\,\hat{E}^{(+)}(t)\rangle}{\langle\hat{E}^{(-)}(t)\,\hat{E}^{(+)}(t)\rangle^2}\,. \tag{4}$$

In the case of a polarized field, the photocurrent is represented by the operator

$$\hat{I}(t)=(ec/L)\hat{E}^{(-)}(t)\,\hat{E}^{(+)}(t)\,, \tag{5}$$

where L is the length of the quantization volume, and the "detection operator" is given by [19]

$$\hat{E}^{(+)}(t)=\sum_k \hat{A}_k\exp(-\mathrm{i}\omega_k t)\,. \tag{6}$$

We emphasize that the photocurrent operator is actually a field operator that has been normalized in such a way that its expectation value represents the photocurrent of an ideal photodetector.

The average photocurrent is calculated from eqs. (5) and (6) to be

$$\langle\hat{I}\rangle=(ec/L)\Big(|g(\omega_0)|^2\langle\hat{a}_0^\dagger\hat{a}_0\rangle+\sum_k\langle\hat{L}_k^\dagger\hat{L}_k\rangle\Big)\,. \tag{7}$$

In eq. (7), the first term represents the amplification of the input signal at the carrier frequency, and the second term represents noise photons that are spontaneously generated by the amplifier. Note that the average output photon number of the vacuum input modes is given by

$$\langle\hat{N}_k\rangle=\langle\hat{A}_k^\dagger\hat{A}_k\rangle=\langle\hat{L}_k^\dagger\hat{L}_k\rangle,\quad k\neq 0\,. \tag{8}$$

In any physical situation, these spontaneously generated photons can occur only over a finite bandwidth. Therefore, for the case in which the number of photons in the excited input mode is sufficiently large [$\langle\hat{a}_0^\dagger\hat{a}_0\rangle\gg 1$], the average photocurrent reduces to good approximation to

$$\langle\hat{I}\rangle=(ec/L)\,|g(\omega_0)|^2\,\langle\hat{a}_0^\dagger\hat{a}_0\rangle\,, \tag{9}$$

as expected of a linear amplifier.

The calculation of the fourth-order correlation function of the detection operator $\langle\hat{E}^{(-)}(t)\,\hat{E}^{(-)}(t+\tau)\,\hat{E}^{(+)}(t+\tau)\,\hat{E}^{(+)}(t)\rangle$ reduces to the calculation of the expectation value

$$\langle\hat{A}_k^\dagger\hat{A}_l^\dagger\hat{A}_m\hat{A}_n\rangle=\langle[g^*(\omega_k)\hat{a}_k^\dagger+\hat{L}_k^\dagger]\,[g^*(\omega_l)\hat{a}_l^\dagger+\hat{L}_l^\dagger]\times[g(\omega_m)\hat{a}_m+\hat{L}_m]\,[g(\omega_n)\hat{a}_n+\hat{L}_n]\rangle\,. \tag{10}$$

To calculate the power spectrum of the fluctuations in the photocurrent, we are only interested in the portion of the correlation function that depends on τ. After the expectation value of eq. (10) is evaluated for the case of large number of input photons, the τ-dependent part of the fourth-order correlation function is given by

$$\langle\hat{E}^{(-)}(t)\,\hat{E}^{(-)}(t+\tau)\,\hat{E}^{(+)}(t+\tau)\,\hat{E}^{(+)}(t)\rangle_{\mathrm{F}}=|g(\omega_0)|^2\,\langle\hat{a}_0^\dagger\hat{a}_0\rangle\sum_k\langle\hat{L}_k^\dagger\hat{L}_k\rangle\times\{\exp[\mathrm{i}(\omega_k-\omega_0)\tau]+\exp[-\mathrm{i}(\omega_k-\omega_0)\tau]\}\,, \tag{11}$$

where the subscript F denotes the fluctuating part. By going to the continuous limit of mode frequencies using the convention

$$\sum_k\to\int_0^\infty \mathrm{d}\omega\,\frac{L}{2\pi c}\,, \tag{12}$$

the correlation function of the photocurrent fluctuations becomes

$$C_{\mathrm{F}}(\tau)=e\langle\hat{I}\rangle\,\delta(\tau)+\frac{e}{2\pi}\langle\hat{I}\rangle\int_0^\infty \mathrm{d}\omega\,\langle\hat{L}_\omega^\dagger\hat{L}_\omega\rangle\times\{\exp[\mathrm{i}(\omega-\omega_0)\tau+\exp[-\mathrm{i}(\omega-\omega_0)\tau]\}\,. \tag{13}$$

The power spectrum of the fluctuations is then given by [20]

$$S_{\mathrm{F}}(\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} C_{\mathrm{F}}(\tau)\exp(\mathrm{i}\Omega\tau)\,\mathrm{d}t=(2\pi)^{-1}e\langle\hat{I}\rangle\,[1+\langle\hat{N}_{\omega_0-\Omega}\rangle+\langle\hat{N}_{\omega_0+\Omega}\rangle]\,. \tag{14}$$

This result is seen to depend on the properties of the amplifier at two frequency sidebands that are displaced by $\pm\Omega$ from the carrier frequency ω_0 of the input field. The first term on the right-hand side in eq. (14) represents the shot noise of the amplified output signal, and the second and third terms represent the number of photons added spontaneously to the lower and upper frequency sidebands as a consequence of the noise inherent to the amplification process. In addition, the result is independent of the particular quantum state of the excited input mode.

3. Comparison to the results of single-mode theory

It is instructive to compare the result given by eq. (14) to the corresponding result obtained for the quantum-mechanical ensemble fluctuations of the photon number of the mode $k=0$ for the case of a quantum-noise-limited amplifier and a coherent-state input. For this case, the Langevin noise operators can be expressed in terms of creation operators as

$$\hat{L}_k = h(\omega_k)\,\hat{b}_k^\dagger\,, \tag{15}$$

where the operator $\hat{b}_k$ operates on the respective vacuum state, and where

$$|h(\omega_k)|^2 = |g(\omega_k)|^2 - 1\,. \tag{16}$$

The variance of the photon-number fluctuations of the excited input mode after amplification is given by

$$\langle(\Delta\hat{N}_0)^2\rangle = |g(\omega_0)|^2\,[2|g(\omega_0)|^2 - 1]\,\langle\hat{n}_0\rangle\,, \tag{17}$$

where $\langle\hat{n}_0\rangle = \langle\hat{a}_0^\dagger\hat{a}_0\rangle$. This result implies that the amplified beam is noisier than a shot-noise limited beam with equal average photon number by a factor

$$X = 2|g(\omega_0)|^2 - 1\,. \tag{18}$$

On the other hand, the power spectrum of the photocurrent for this quantum-noise-limited case becomes

$$S_F(\Omega) = (2\pi)^{-1} e\langle\hat{I}\rangle\,[1 + |h(\omega_0 - \Omega)|^2 + |h(\omega_0 + \Omega)|^2]\,, \tag{19}$$

which implies that the spectrum appears noisier than a shot-noise-limited beam with equal intensity by a factor

$$X_F(\Omega) = 1 + |h(\omega_0 - \Omega)|^2 + |h(\omega_0 + \Omega)|^2\,. \tag{20}$$

The results given by eqs. (18) and (20) are found to agree only for the case where the gains at the two frequency sidebands are equal to that at the carrier frequency, i.e., $|h(\omega_0 \pm \Omega)|^2 \approx |h(\omega_0)|^2 = |g(\omega_0)|^2 - 1$. In the case in which at least one of the frequency sidebands experiences larger gain than that experienced by the mode at the carrier frequency, the amplifier will appear much noisier than the single-mode quantum-mechanical limit given by eq. (18). This situation corresponds to amplifying the vacuum fluctuations of the input field at the sideband frequency rather than the input signal itself. On the other hand, if there is no gain at the two frequency sidebands, i.e., $|h(\omega_0 - \Omega)|^2 = |h(\omega_0 + \Omega)|^2 \approx 0$, the fluctuations of the amplified output beam correspond to those of a shot-noise-limited beam.

4. Conclusions

We have derived a general expression for the power spectrum of the output from a quantum amplifier with frequency-dependent gain for the case in which the input mode at the carrier frequency of the field is in an arbitrary quantum state and the other input modes are in the vacuum state. The results depend on the shot-noise level associated with the amplified input mode and on the number of noise photons added to the field by the amplifier at the upper and lower frequency sidebands that are displaced by the spectrum-analyzer frequency from the carrier frequency of the field. For the special case of a quantum-noise-limited amplifier and a coherent-state input field, the results agree with the single-mode results only for the case where the gain of the amplifier at the frequency sidebands is equal to the gain at the carrier frequency of the field.

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