Multidimensional coupling owing to optical nonlinearities. I. General formulation

Robert E. Bridges,* Robert W. Boyd, and Govind P. Agrawal

The Institute of Optics, University of Rochester, Rochester, New York 14627

Received January 23, 1995; revised manuscript received August 15, 1995

A beam propagating in a nonlinear, dispersive medium can experience nonlinear coupling among its parameters in the x, y, and t dimensions. We derive two new computational techniques that account for this coupling. The first technique uses the beam-propagation method to solve three coupled differential equations, and the second uses six coupled second-moment equations. Nonlinear coupling is especially important for the operation of self-mode-locked (Kerr-lens mode-locked) lasers, an example of which is considered in detail in an accompanying paper [J. Opt. Soc. Am. **13,** 560 (1996)]. The two techniques derived here can be adapted to handle most of the physical effects of importance to self-mode-locked lasers, but we concentrate on the effects of diffraction, second- and third-order dispersion, and Kerr nonlinearity.

1. INTRODUCTION

Self-mode-locked (Kerr-lens mode-locked) lasers have developed rapidly since their discovery in 1990.[1] They have produced the shortest pulses of any laser (≈8.5 fs with Ti:sapphire).[2] They have been built from a variety of materials, for example, Nd:YAG,[3] Nd:YLF,[4] Cr:LiSAF,[5] and Cr:YAG,[6] and have been pumped by a variety of sources.

The design of self-mode-locked laser cavities has been investigated in several different ways. The effect of the Kerr nonlinearity on cavity behavior has been studied by use of techniques such as quadratic approximation,[7] change of variables,[8] minimization of mean-squared error,[9] and self-similar solution.[10] The effects of gain[11–13] and of thermal lensing[3,14] on cavity behavior have also been studied.

However, an effect that has not been included in these studies is that of nonlinear coupling among beam parameters in the x and the y dimensions. This effect is important because the X- and Z-shaped cavities[15,16] of self-mode-locked lasers give rise to elliptical, rather than circular, beams. When nonlinearities that depend on intensity (e.g., Kerr nonlinearity, saturated gain, thermal lensing) are present, elliptical beams evolve in a way that depends on the beam widths in both the x and the y directions. It is not possible, therefore, to calculate the evolution of the beam in the x direction without also considering the evolution in the y direction and vice versa.

Previously we reported a technique that generalizes the methods of Refs. 7–10 to include the effects of nonlinear x–y coupling.[17] We demonstrated that nonlinear coupling can have a significant effect on the self-mode-locking process. In some cases, nonlinear coupling completely changes the conditions under which self-mode locking is achieved.

More recently, an approach has been developed to calculate analytically the self-consistent (Gaussian) solution in the astigmatic cavity of a self-mode-locked laser.[18] This approach is based on the assumption that the Kerr nonlinearity causes a small fractional change in beam size throughout the cavity. Consequently, the technique is most accurate in the small-power limit.

In addition to the spatial characteristics of self-mode-locked lasers, the temporal characteristics have also been investigated. Techniques used to model temporal behavior include master equations,[19] self-consistent solution of lumped, noncommuting operations,[20] iterative integral equations,[21] and coupled spatial–temporal analysis.[22] The first three techniques neglect the coupling among spatial (x, y) and temporal (t) beam parameters. This coupling can be important because beam area affects intensity, which affects the evolution in the t dimension. Similarly, pulse width affects intensity, which affects evolution in the x and y dimensions. The fourth technique (Ref. 22) does account, to some extent, for this space–time coupling, but it does not consider coupling between the x and the y dimensions.

The effects of nonlinear coupling are also important outside the context of the self-mode-locked laser. Previously these effects were investigated for the case of an unconfined beam propagating in a Kerr medium with coupling in two dimensions (x and y) but not in three dimensions (x, y, and t). The numerical techniques that have been used include fast Fourier transform/Runge–Kutta,[23] second moments,[17] self-similar solution,[24] and WKB approximation.[25]

In this paper we develop numerical techniques for calculating beam evolution in all three dimensions, x, y, and t, while taking into account the effects of nonlinear coupling. The accompanying paper[26] applies these techniques to a self-mode-locked laser and compares the accuracies that are obtained.

For the type of problem considered here, the most accurate calculations are based on three-dimensional numerical methods, for example, a beam propagation (split-step) method that uses three-dimensional fast Fourier transforms (3D FFT's). Such techniques have not been applied to a thorough treatment of self-mode-locked lasers, however, because of the vast computing resources that

0740-3224/96/030553-07$06.00

would be required. We show that a much faster numerical method, making use of one-dimensional (1D) FFT's to solve three coupled differential equations, can be developed if the field amplitude is separable in x, y, and t. We demonstrate that an even faster method that uses coupled second-moment equations can be obtained if the spatial and temporal profiles of the beam do not change significantly during propagation.

The numerical methods discussed in this paper can be adapted to handle most physical effects of importance to self-mode-locked lasers. These effects include diffraction, second-, third-, and fourth-order dispersion, Kerr nonlinearity, loss, saturated gain, thermal lensing, and gain dispersion. This paper, however, concentrates primarily on diffraction, second- and third-order dispersion, and Kerr nonlinearity.

2. GENERALIZED PROPAGATION EQUATION

The optical field inside a laser cavity results from a self-consistent solution of Maxwell's equations after all the variations in gain, loss, and index of refraction are accounted for. Using Maxwell's equations, together with the constitutive relations, we proceed in the usual way[27] to obtain (in SI units)

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, \omega) - \epsilon_L(\omega) \frac{\omega^2}{c^2} \mathbf{E}(\mathbf{r}, \omega) = \frac{\omega^2}{\epsilon_0 c^2} \mathbf{P}_{\mathrm{NL}}(\mathbf{r}, \omega). \tag{1}$$

Here $\mathbf{E}(\mathbf{r}, \omega)$ is the electric field, $\mathbf{r} = (x, y, z)$ is the position vector, z is the propagation distance, ω is the temporal frequency, c is the speed of light, ϵ_0 is the permittivity of free space, $\epsilon_L(\omega)$ is the linear relative permittivity, and $\mathbf{P}_{\mathrm{NL}}(\mathbf{r}, \omega)$ is the nonlinear polarization.

In a self-mode-locked laser it is usually necessary for the electric field to maintain a linear, nonrotating polarization to avoid frequency-dependent losses. Consequently the vector quantities $\mathbf{E}(\mathbf{r}, \omega)$ and $\mathbf{P}_{\mathrm{NL}}(\mathbf{r}, \omega)$ can be replaced with the corresponding scalar quantities, $E(\mathbf{r}, \omega)$ and $P_{\mathrm{NL}}(\mathbf{r}, \omega) = \epsilon_0 \epsilon_{\mathrm{NL}}(\mathbf{r}, \omega) E(\mathbf{r}, \omega)$. We define the total relative permittivity as $\epsilon = \epsilon_L + \epsilon_{\mathrm{NL}}$, the index of refraction as $n(\mathbf{r}, \omega) = \sqrt{\epsilon(\mathbf{r}, \omega)}$, and the propagation constant as $\beta(\mathbf{r}, \omega) = n(\mathbf{r}, \omega)\omega/c$. The leftmost term in Eq. (1) can be approximated as $-\nabla^2 \mathbf{E}(\mathbf{r}, \omega)$ because $|\nabla(\nabla \cdot \mathbf{E})| \ll |\nabla^2 \mathbf{E}|$. With the above substitutions, Eq. (1) becomes

$$\nabla^2 E(\mathbf{r}, \omega) + \beta^2(\mathbf{r}, \omega) E(\mathbf{r}, \omega) = 0. \tag{2}$$

We define the electric field amplitude $A(\mathbf{r}, T)$ by the relation

$$E(\mathbf{r}, T) = {}^1\!/\!{}_2 A(\mathbf{r}, T)\exp(i\beta_0 z - i\omega_0 T) + \text{c.c.}, \tag{3a}$$

where $\beta_0 = n_0 \omega_0 / c$ is the linear propagation constant, n_0 is the linear index of refraction at the reference frequency ω_0, and T is time in the laboratory frame of reference. We take the Fourier transform of each side of this equation, then use the slowly varying envelope approximation to drop one term. The result is

$$E(\mathbf{r}, \omega) \approx {}^1\!/\!{}_2 A(\mathbf{r}, \omega - \omega_0)\exp(i\beta_0 z). \tag{3b}$$

We substitute this expression into Eq. (2), then apply the approximation $\beta_0^2 - \beta^2 \approx 2\beta_0(\beta_0 - \beta)$. Finally, we use the paraxial (slowly varying amplitude) approximation to drop the $\partial^2 A/\partial z^2$ term. The result is

$$i \frac{\partial}{\partial z} A = -\frac{1}{2\beta_0} \frac{\partial^2}{\partial x^2} A - \frac{1}{2\beta_0} \frac{\partial^2}{\partial y^2} A - \frac{\omega_0}{c}(n - n_0)A. \tag{4}$$

The (complex) change in the index of refraction at some point in the cavity is

$$n - n_0 \approx \Delta n_K + \Delta n_T + \Delta n_L - i \frac{c}{\omega_0} \frac{g - \alpha}{2}. \tag{5}$$

Here Δn_K is the change in the index that is due to the Kerr nonlinearity, Δn_T is the change in the index that is due to thermal nonlinearity, and Δn_L is the change in the index that is due to dispersion:

$$\Delta n_L(\omega) \approx \frac{c}{\omega_0}\left[\beta_1(\omega - \omega_0) + \frac{\beta_2}{2}(\omega - \omega_0)^2 + \frac{\beta_3}{6}(\omega - \omega_0)^3 + \cdots\right]. \tag{6}$$

The quantity $\beta_m = \left(\mathrm{d}^m \beta(\omega)/\mathrm{d}\omega^m\right)_{\omega=\omega_0}$ is the mth-order dispersion parameter, and $\beta(\omega) = n(\omega)\omega/c$ is the propagation constant. The quantities β_2 and β_3 are often called the group-velocity-dispersion and cubic-dispersion parameters, respectively. The saturated gain coefficient for an active medium modeled as a homogeneously broadened two-level system is[28]

$$g(x, y; z) = \frac{g_0(x, y; z)}{1 + I_A(x, y; z)/I_S}, \tag{7}$$

where g_0 is the unsaturated (small-signal) gain, I_A is the time-averaged intensity that accounts for both the forward- and the backward-propagating beams, and I_S is the saturation intensity. In this expression $g(x, y; z)$ is taken to be real because the main effect of the imaginary part of $g(x, y; z)$ is to produce a shift in the frequency of oscillation away from the resonant frequency of the empty, passive cavity.[29] For the problem treated in this paper the slight change in the frequency of oscillation is not of concern. The loss coefficient, which is assumed to be linear, is given by α.

We substitute approximations (5) and (6) into Eq. (4). We take the inverse Fourier transform of all terms, which causes each factor of $\omega - \omega_0$ to be replaced by $i\partial/\partial T$. Finally, we eliminate the term containing β_1 by making the change of variables $t = T - z/v_g$, where t is the time referenced to the center of the pulse and $v_g = 1/\beta_1$ is the group velocity. The result of these operations is a generalized propagation equation:

$$i \frac{\partial}{\partial z} A = (\hat{D}_x + \hat{D}_y + \hat{D}_t + \hat{D}_t^{(3)} + \hat{K} + \hat{G} + \hat{L} + \hat{T} + \cdots)A. \tag{8}$$

Here the operators for diffraction in x and y and dispersion in t are

$$\hat{D}_x = -\frac{1}{2\beta_0}\frac{\partial^2}{\partial x^2}, \qquad \hat{D}_y = -\frac{1}{2\beta_0}\frac{\partial^2}{\partial y^2},$$

$$\hat{D}_t = \frac{\beta_2}{2}\frac{\partial^2}{\partial t^2}, \qquad \hat{D}_t^{(3)} = i\frac{\beta_3}{6}\frac{\partial^3}{\partial t^3}. \tag{9}$$

The operators for Kerr nonlinearity, saturated gain, loss, and thermal lensing are $\hat{K} = -(\omega_0 n_2'/2c)|A|^2$, $\hat{G} = ig/2$, $\hat{L} = -i\alpha/2$, and $\hat{T} = -\omega_0 \Delta n_T / c$, respectively. The nonlinearity parameter n_2' is defined by the relation $\Delta n_K = {}^1\!/\!{}_2 n_2' |A|^2$ and is a constant that depends on the particular material. Other physical effects can also be important. Examples are fourth-order dispersion, which is especially important for the shortest-pulse lasers (~10 fs),[21] and gain dispersion, which accounts for the variation in gain and loss as a function of wavelength.

3. FOURIER-TRANSFORM TECHNIQUES

The beam propagation method (BPM) is useful when diffraction, dispersion, and nonlinearities are present.[27] The distinctive feature of this method is that calculations for nonlinearities are performed independently of calculations for diffraction and dispersion.

To apply the method, we first consider diffraction and dispersion alone. For this case, Eq. (8) reduces to $i(\partial/\partial z)A = \hat{D}A$, where $\hat{D}(x, y, t) = \hat{D}_x + \hat{D}_y + \hat{D}_t + \hat{D}_t^{(3)}$. We take the Fourier transform of each side of this equation, solve the resulting differential equation for a step size h, and then take the inverse Fourier transform. The result is

$$A(x, y, t; z + h) = \mathcal{D}A(x, y, t; z), \tag{10a}$$

where

$$\hat{\mathcal{D}} = \hat{\mathcal{F}}_{xyt}^{-1} \exp[-ih\hat{D}(k_x, k_y, \omega - \omega_0)]\hat{\mathcal{F}}_{xyt}. \tag{10b}$$

Here $\hat{\mathcal{F}}_{xyt}$ is the three-dimensional Fourier-transform operator and

$$\begin{aligned}\hat{D}(k_x, k_y, \omega - \omega_0) &= \hat{\mathcal{F}}_{xyt}[D(x, y, t)] \\ &= \frac{1}{2\beta_0} k_x^2 + \frac{1}{2\beta_0} k_y^2 - \frac{\beta_2}{2}(\omega - \omega_0)^2 \\ &\quad - \frac{\beta_3}{6}(\omega - \omega_0)^3.\end{aligned} \tag{11}$$

Next, the nonlinearities alone are considered. For this case Eq. (8) reduces to $i(\partial/\partial z)A = \hat{N}A$, where $\hat{N} = \hat{N}(x, y, t; z) = \hat{K} + \hat{T} + \hat{L} + \hat{G} + \cdots$. We solve the differential equation to get $A(x, y, t; z + h) = \hat{\mathcal{N}}A(x, y, t; z)$, where $\hat{\mathcal{N}} = \exp[-i\int_z^{z+h}\hat{N}(z')\mathrm{d}z']$. To obtain numerical results we make an estimate of this last integral.

It is possible to combine the effects of diffraction and dispersion with the effects of nonlinearity by simply multiplying the $\hat{\mathcal{D}}$ and $\hat{\mathcal{N}}$ operators to obtain $A(x, y, t; z + h) = \hat{\mathcal{N}}\hat{\mathcal{D}}A(x, y, t; z)$, but the results are accurate only to the second order in step size h. It is better to use the symmetrized form,[27] $A(x, y, t; z + h) = \hat{\mathcal{D}}_{1/2}\hat{\mathcal{N}}\hat{\mathcal{D}}_{1/2}A(x, y, t; z)$, where $\hat{\mathcal{D}}_{1/2} = \hat{\mathcal{F}}_{xyt}^{-1}\exp[-i\tfrac{1}{2}h\hat{D}(k_x, k_y, \omega - \omega_0)]\hat{\mathcal{F}}_{xyt}$, as this gives results accurate to the third order in h.

For the general case, we must use the BPM with 3D FFT's. However, if the field amplitude is separable in x, y, and t such that

$$A(x, y, t; z) = B_x(x; z)B_y(y; z)B_t(t; z), \tag{12}$$

then we can derive a much faster numerical method that uses 1D FFT's to solve coupled differential equations.

Before deriving these coupled differential equations, we check to see whether the assumption of Eq. (12) seems reasonable. First, we neglect the nonlinearity and substitute Eq. (12) into Eqs. (10) to obtain

$$A(x, y, t; z + h) = B_x(x; z + h)B_y(y; z + h)B_t(t; z + h), \tag{13}$$

where

$$B_x(x; z + h) = \hat{\mathcal{F}}_x^{-1}\exp\left(-i\frac{hk_x^2}{2\beta_0}\right)\hat{\mathcal{F}}_x B_x(x; z), \tag{14a}$$

$$B_y(y; z + h) = \hat{\mathcal{F}}_y^{-1}\exp\left(-i\frac{hk_y^2}{2\beta_0}\right)\hat{\mathcal{F}}_y B_y(y; z), \tag{14b}$$

$$\begin{aligned}B_t(t; z + h) = \hat{\mathcal{F}}_t^{-1}\exp\bigg[&i\frac{h\beta_2(\omega - \omega_0)^2}{2} \\ &+ i\frac{h\beta_3(\omega - \omega_0)^3}{6}\bigg]\hat{\mathcal{F}}_t B_t(t; z).\end{aligned} \tag{14c}$$

Here, $\hat{\mathcal{F}}_x$, $\hat{\mathcal{F}}_y$, and $\hat{\mathcal{F}}_t$ are 1D FFT operators. Comparing Eqs. (12) and (13), we see that, in a linear system, a field that is separable will remain separable for any propagation distance h.

Next we check what happens when we include the Kerr nonlinearity. The nonlinear operator $\hat{\mathcal{N}}$ is given by $\hat{\mathcal{N}} = \exp[(i\omega_0 n_2' h/2c)|B_x|^2|B_y|^2|B_t|^2]$. It is easy to see that $\hat{\mathcal{N}}$ is not separable in x, y, and t, even if B_x, B_y, and B_t begin as Gaussian functions. So, in general, Eq. (12) will not be true when nonlinearity is present. However, in practical situations, Eq. (12) is often a good approximation. We demonstrate this in the accompanying paper[26] by considering a specific example of a pulse propagating in a nonlinear medium and comparing the numerical results obtained with and without the assumption of Eq. (12).

We now assume that Eq. (12) is valid and proceed to derive the coupled differential equations. We define a normalized amplitude, $u(x, y, t; z) = A(x, y, t; z)/\sqrt{M(z)}$, where $M(z) = \iiint_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}y\mathrm{d}t\,|A(x, y, t; z)|^2$. Then, following Eq. (12), we can write $u(x, y, t; z) = u_x(x; z)u_y(y; z)u_t(t; z)$. We note that $\iiint_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}y\mathrm{d}t|u|^2 = 1$, so it is consistent to require that $\int_{-\infty}^{\infty}\mathrm{d}x|u_x|^2 = 1$, $\int_{-\infty}^{\infty}\mathrm{d}y|u_y|^2 = 1$, and $\int_{-\infty}^{\infty}\mathrm{d}t|u_t|^2 = 1$.

To account for the effects of saturated gain and thermal lensing in Eq. (8), it is possible to extend the computational techniques presented in this paper, but this requires considerable additional development. Consequently, here we drop the terms that account for gain, loss, and thermal lensing to obtain $i\partial A/\partial z = (\hat{D}_x + \hat{D}_y + \hat{D}_t + \hat{D}_t^{(3)} + \hat{K})A$. We substitute $A = \sqrt{M}\,u_x u_y u_t$ into this equation, then apply the operator $(1/u_x)\iint_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}t u_y^* u_t^*$ to each term. After simplifying, we are able to write

$$i\frac{1}{u_x}\frac{\partial u_x}{\partial z} + \frac{1}{2\beta_0}\frac{1}{u_x}\frac{\partial^2 u_x}{\partial x^2} + \frac{\omega_0 n_2'}{2c}\frac{M}{\delta_y\delta_t}|u_x|^2 = c_1(z), \tag{15}$$

where $c_1(z) = -i\int_{-\infty}^{\infty}\mathrm{d}y u_y^*(\partial u_y/\partial z) - (1/2\beta_0)\int_{-\infty}^{\infty}\mathrm{d}y u_y^*(\partial^2 u_y/\partial y^2)$ is a real function of z only, $\delta_y = (\int_{-\infty}^{\infty}|u_y|^4\mathrm{d}y)^{-1}$ is the effective beam width in the y direction, and $\delta_t = (\int_{-\infty}^{\infty}|u_t|^4\mathrm{d}t)^{-1}$ is the effective pulse width. Applying the BPM to Eq. (15), we see that the physical effect of the constant $c_1(z)$ is to cause a longitudinal phase shift in the field amplitude u_x that is a function of z but not of x, y, or t. In this paper we are not concerned with this longitudinal phase shift and therefore make the simplifying assumption (which is precisely valid for the linear case but not for the nonlinear case) that $c_1(z) = 0$.

To understand the physical meaning of the quantity M in Eq. (15), we note that, with SI units and the conventions of this paper, $|A(x, y, t; z)|^2 = 2I(x, y, t; z)/n_0\epsilon_0 c$,[30] where I is the intensity. Then $M(z) = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}t|A|^2 = 2U/n_0\epsilon_0 c$, where U is the pulse energy. The coefficient n_2, defined by the relation $\Delta n_K = n_2 I$, is related to the coefficient n_2' by $n_2' = n_0\epsilon_0 c n_2$, from which it follows that $n_2' M = 2n_2 U$. We use this, along with the definition of the critical power, $P_c = 2\pi n_0/\beta_0^2 n_2$,[31,32] to rewrite the nonlinear term in Eq. (15) as $(2\pi U/\beta_0\delta_y\delta_t P_c)|u_x|^2$. (Note that P_c can be negative as defined.) The physical meaning of the critical power will become clear below. Repeating the process above for the y and t dimensions, we find the three coupled equations:

$$i\,\frac{\partial u_x}{\partial z} = -\frac{1}{2\beta_0}\,\frac{\partial^2 u_x}{\partial x^2} - \frac{2\pi}{\beta_0\delta_y\delta_t}\,\frac{U}{P_c}|u_x|^2 u_x\,, \tag{16a}$$

$$i\,\frac{\partial u_y}{\partial z} = -\frac{1}{2\beta_0}\,\frac{\partial^2 u_y}{\partial y^2} - \frac{2\pi}{\beta_0\delta_x\delta_t}\,\frac{U}{P_c}|u_y|^2 u_y\,, \tag{16b}$$

$$i\,\frac{\partial u_t}{\partial z} = \frac{\beta_2}{2}\,\frac{\partial^2 u_t}{\partial t^2} + i\,\frac{\beta_3}{6}\,\frac{\partial^3 u_t}{\partial t^3} - \frac{2\pi}{\beta_0\delta_x\delta_y}\,\frac{U}{P_c}|u_t|^2 u_t\,. \tag{16c}$$

We solve these equations by using the BPM with 1D FFT's, as described above. The three equations above are coupled to one another through the effective widths δ_x, δ_y, and δ_t. From a computational point of view, the implication of nonlinear coupling is that, each time a propagation step is taken, all three of Eqs. (16a)–(16c) must be solved before we move on to the next step.

For a continuous-wave beam we can derive two differential equations to replace the three equations above. Similarly, if the pulse width is so long that it does not change its value significantly during propagation, then δ_t is approximately constant, and the effective power can be defined as $P = U/\delta_t$. In either case, the resulting equations are

$$i\,\frac{\partial u_x}{\partial z} = -\frac{1}{2\beta_0}\,\frac{\partial^2 u_x}{\partial x^2} - \frac{2\pi}{\beta_0\delta_y}\,\frac{P}{P_c}|u_x|^2 u_x\,, \tag{17a}$$

$$i\,\frac{\partial u_y}{\partial z} = -\frac{1}{2\beta_0}\,\frac{\partial^2 u_y}{\partial y^2} - \frac{2\pi}{\beta_0\delta_x}\,\frac{P}{P_c}|u_y|^2 u_y\,. \tag{17b}$$

Similar equations can be derived for the two dimensions x and t; these are useful for the propagation of an optical pulse in a planar waveguide. Also, equations can be derived in one dimension for x, y, or t. For example, we can use Eq. (16c) along with the definition $\mathcal{U}(t; z) = u_t(t; z)/u_t(0; 0)$ to obtain a well known equation for the propagation of a pulse in an optical fiber[33]:

$$i\,\frac{\partial \mathcal{U}}{\partial z} = \frac{\beta_2}{2}\,\frac{\partial^2 \mathcal{U}}{\partial t^2} + i\,\frac{\beta_3}{6}\,\frac{\partial^3 \mathcal{U}}{\partial t^3} - \gamma P_0|\mathcal{U}|^2\mathcal{U}\,, \tag{18}$$

where $P_0 = U|u_t(0; 0)|^2$ is the initial power at the center of the pulse, $\gamma = 2\pi/\beta_0 A_e P_c = n_2\omega_0/cA_e$ is a coefficient of the Kerr nonlinearity, and $A_e = \delta_x\delta_y$ is the effective area of the optical fiber.

4. EQUATIONS BASED ON SECOND MOMENTS

Numerical methods using Fourier transforms give detailed information about the field amplitude in the x, y, and t dimensions. In contrast, equations based on moments with respect to the intensity distribution give information only about the average properties of the beam. In particular, second-moment methods give information about the spatial and temporal widths of the beam and the rate of change of these widths with propagation distance z. In many cases the beam profiles in the x, y, and t dimensions are relatively smooth and hence well described by these average properties. In these cases one would like to use methods based on second moments, if possible, because of the greatly reduced computing time.

For problems involving nonlinear cavities it is necessary to bounce the beam back and forth within the cavity many times to find the self-consistent cavity solution. We can speed the computations considerably by using second-moment equations (SME's) to get close to the solution. Following this, an FFT technique can be used, if desired, to improve accuracy.

It is possible to use SME's to account for third-order dispersion in a linear cavity when the profile of the beam is Gaussian.[34] In the more general case, however, the SME's are less successful in accounting for the third-order dispersion, and an FFT method is preferable. It is possible to use the SME's to account for thermal lensing and gain saturation, but we do not treat these cases here because of space limitations. Our starting point for the development of the coupled SME's is therefore Eqs. (16) but with the third-order dispersion term removed from Eq. (16c). If we let the subscript ν represent x, y, or t, we can write these equations in the general form

$$i\partial u_\nu/\partial z = (\hat{D}_\nu + \hat{K}_\nu)u_\nu\,. \tag{19}$$

Here the dispersion-diffraction operator $\hat{D}_\nu$ is defined as

$$\hat{D}_\nu = -\frac{d_\nu}{2}\,\frac{\partial^2}{\partial \nu^2}\,, \tag{20}$$

where d_ν is the dispersion-diffraction parameter, given for $\nu = x$, y, or t by

$$d_x = d_y = 1/\beta_0, \qquad d_t = -\beta_2\,. \tag{21}$$

The Kerr operator $\hat{K}_\nu$ is defined as $\hat{K}_\nu = -\kappa_\nu|u_\nu|^2$, where $\kappa_\nu = 2\pi U/\beta_0\Pi_\nu P_c$ is the Kerr coefficient and $\Pi_x = \delta_y\delta_t$, $\Pi_y = \delta_x\delta_t$, and $\Pi_t = \delta_x\delta_y$ are the effective space–time areas.

The definition of the second moment of the width with respect to the intensity distribution (also known as the mean-squared width) in the dimension ν is $\rho_\nu^2 = \langle\nu^2\rangle - \langle\nu\rangle^2$, where $\langle\nu^2\rangle = \int_{-\infty}^{\infty}\nu^2|u_\nu|^2\mathrm{d}\nu$ and $\langle\nu\rangle = \int_{-\infty}^{\infty}\nu|u_\nu|^2\mathrm{d}\nu$. If only the physical effects of Eq. (19) are considered, and if $\langle\nu\rangle = 0$ at $z = 0$, then $\langle\nu\rangle = 0$ for all z. The second moment can therefore be written as $\rho_\nu^2 = \int_{-\infty}^{\infty}\nu^2|u_\nu|^2\mathrm{d}\nu$. The rms width of the beam in the ν dimension, ρ_ν, is the square root of the second moment.

The first step in deriving the coupled SME's is to find a way to calculate $\partial\rho_\nu^2/\partial z$ in terms of the transverse variables x, y, and t. We manipulate the basic relation $\partial\rho_\nu^2/\partial z = (\partial/\partial z)\int_{-\infty}^{\infty}\nu^2|u_\nu|^2\mathrm{d}\nu$ to obtain $\partial\rho_\nu^2/\partial z =$

$\hat{a}^{(1)}(i\partial u_\nu/\partial z)$, where the operator $\hat{a}^{(1)}$ is defined as $\hat{a}^{(1)} = 2\,\mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}\nu\nu^2 u_\nu^*$. Inasmuch as $i\partial u_\nu/\partial z$ is the leftmost term in Eq. (19), it follows that $\partial\rho_\nu^2/\partial z = \hat{a}^{(1)}(\hat{D}_\nu + \hat{K}_\nu)u_\nu$. Carrying out these operations and using integration by parts yields

$$\frac{\partial\rho_\nu^2}{\partial z} = 2d_\nu\ \mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}\nu\nu u_\nu^*\frac{\partial u_\nu}{\partial\nu} \equiv -d_\nu C_\nu\,. \qquad (22)$$

The quantity C_ν is known as the chirp parameter. We will discuss its physical meaning presently.

The next step in deriving the SME's is to find an expression for $\partial^2\rho_\nu^2/\partial z^2$. To do this we manipulate Eq. (22) to obtain $\partial^2\rho_\nu^2/\partial z^2 = (\hat{a}_1^{(2)} + \hat{a}_2^{(2)})(i\partial u_\nu/\partial z)$, where $\hat{a}_1^{(2)} = 2d_\nu\ \mathrm{Re}\int_{-\infty}^{\infty}\mathrm{d}\nu\nu(\partial u_\nu/\partial\nu)\hat{\Xi}$ and $\hat{a}_2^{(2)} = -2d_\nu\ \mathrm{Re}\int_{-\infty}^{\infty}\mathrm{d}\nu\nu\, u_\nu^*(\partial/\partial\nu)$. Here $\hat{\Xi}$ is an operator that takes the complex conjugate of its argument. From Eq. (19) it follows that $\partial^2\rho_\nu^2/\partial z^2 = (\hat{a}_1^{(2)} + \hat{a}_2^{(2)})(\hat{D}_\nu + \hat{K}_\nu)u_\nu$. To carry out the operations in this equation we use techniques such as integration by parts, expansion of the derivative terms, and use of real and imaginary parts to eliminate terms. The result is

$$\frac{\partial^2\rho_\nu^2}{\partial z^2} = 2d_\nu^2\int_{-\infty}^{\infty}\mathrm{d}\nu\left|\frac{\partial u_\nu}{\partial\nu}\right|^2 - d_\nu\kappa_\nu\int_{-\infty}^{\infty}\mathrm{d}\nu|u_\nu|^4\,. \qquad (23)$$

We would like to rewrite this equation, using physical quantities that are easier to interpret. To do this we expand the normalized field amplitude as

$$u_\nu(\nu;z) = f_\nu(\nu;z)\exp\left(-i\frac{C_\nu\nu^2}{4\rho_\nu^2}\right), \qquad (24)$$

where $f_\nu(\nu;z)$ is the chirp-free normalized field amplitude. The function f_ν must satisfy the relation $\mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}\nu f_\nu^*\partial f_\nu/\partial\nu = 0$ for Eq. (22) to be true. This relation is satisfied automatically if f_ν is real.

To extend the physical interpretation to account for non-Gaussian beam (pulse) profiles we define two additional shape-factor parameters, σ_ν^2 and η_ν. The first of these is

$$\sigma_\nu^2 = \frac{\int_{-\infty}^{\infty}\mathrm{d}\nu|\partial f_\nu/\partial\nu|^2}{\int_{-\infty}^{\infty}\mathrm{d}\nu|\partial f_{g\nu}/\partial\nu|^2} = 4\rho_\nu^2\int_{-\infty}^{\infty}\mathrm{d}\nu\left|\frac{\partial f_\nu}{\partial\nu}\right|^2, \qquad (25)$$

where $f_{g\nu} = \left(2\pi\rho_\nu^2\right)^{-1/4}\exp\left(-\nu^2/4\rho_\nu^2\right)$ is the chirp-free Gaussian amplitude. The factor σ_ν, which is just the square root of the quantity above, is known as the times-diffraction-limit number, the beam-quality factor, or the M^2 factor.[35–39] We show below that this quantity is related to the far-field diffraction angle of a beam propagating in a linear medium. For a Gaussian profile the beam-quality factor takes on the smallest possible value, $\sigma_\nu = 1$. For a hyperbolic-secant profile, $\sigma_\nu = \pi/3$.

The nonlinear shape factor η_ν is defined as

$$\eta_\nu = \frac{\int_{-\infty}^{\infty}|u_\nu|^4\,\mathrm{d}\nu}{\int_{-\infty}^{\infty}|u_{g\nu}|^4\,\mathrm{d}\nu} = 2\sqrt{\pi}\,\rho_\nu\int_{-\infty}^{\infty}|u_\nu|^4\mathrm{d}\nu = \frac{2\sqrt{\pi}\,\rho_\nu}{\delta_\nu}, \qquad (26)$$

where $u_{g\nu} = (2\pi\rho_\nu^2)^{-1/4}\exp(-\nu^2/4\rho_\nu^2)\exp(-iC_\nu\nu^2/4\rho_\nu^2)$ is the normalized Gaussian field amplitude and $\delta_\nu = (\int_{-\infty}^{\infty}\mathrm{d}\nu|u_\nu|^4)^{-1}$ is the effective width. The nonlinear shape factor is equal to 1 for a Gaussian beam and may be either less than or greater than 1 for other beam (pulse) profiles. For a hyperbolic-secant profile, $\eta_\nu = (\pi/3)^{3/2}$.

We substitute Eq. (24) into Eq. (23) and use Eqs. (25) and (26), along with the condition $\mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}\nu f_\nu^*\partial f_\nu/\partial\nu = 0$, to obtain

$$\frac{\partial^2\rho_\nu^2}{\partial z^2} = \frac{d_\nu^2}{2\rho_\nu^2}\left(\sigma_\nu^2 + C_\nu^2 - \frac{\epsilon_\nu^{(3)}\eta_{xyt}}{d_\nu\beta_0}\frac{U}{P_c}\right), \qquad (27)$$

where $\eta_{xyt} = \eta_x\eta_y\eta_t$ is the three-dimensional nonlinear shape factor and $\epsilon_\nu^{(3)} = \rho_\nu^2/2\sqrt{\pi}\rho_x\rho_y\rho_t$ is the three-dimensional ellipticity factor.

Using techniques like those employed to find the first and second derivatives, $\partial\rho_\nu^2/\partial z$ and $\partial^2\rho_\nu^2/\partial z^2$, of Eqs. (22) and (27), we find that the higher-order derivatives, $\partial^m\rho_\nu^2/\partial z^m$ for $m \ge 3$, are in general nonzero. To expand $\rho_\nu^2(z)$ in a Taylor series about an initial position z_1, therefore, we require many terms. If, however, we make the step size $h = z - z_1$ small enough, then the terms of order h^3 and higher become negligible, and we can write

$$\rho_\nu^2(z_1+h) = \rho_\nu^2(z_1) + (\partial\rho_\nu^2/\partial z)_{z=z_1}h + 1/2(\partial^2\rho_\nu^2/\partial z^2)_{z=z_1}h^2\,.$$

We substitute Eqs. (22) and (27) into this equation to obtain

$$\rho_\nu^2(z_1+h) = \rho_{\nu 1}^2\left[\left(1 - \frac{C_{\nu 1}h}{z_{d\nu 1}}\right)^2 + \left(\frac{h}{z_{d\nu 1}}\right)^2 \times\left(\sigma_\nu^2 - \frac{\epsilon_\nu^{(3)}\eta_{xyt}}{d_\nu\beta_0}\frac{U}{P_c}\right)\right], \qquad (28a)$$

where ν equals x, y, or t, $z_{d\nu} = 2\rho_\nu^2/d_\nu$ is the dispersion-diffraction distance (which may be negative), and the subscript 1 indicates that a parameter is evaluated at the initial position z_1. We obtain a set of equations for the chirp parameters C_ν by taking the derivative with respect to h of each term in Eq. (28a) and using Eq. (22). The result is

$$C_\nu(z_1+h) = C_{\nu 1}\left(1 - \frac{C_{\nu 1}h}{z_{d\nu 1}}\right) - \frac{h}{z_{d\nu 1}}\left(\sigma_\nu^2 - \frac{\epsilon_\nu^{(3)}\eta_{xyt}}{d_\nu\beta_0}\frac{U}{P_c}\right). \qquad (28b)$$

To use Eqs. (28) we must know the values of five quantities at the beginning of the step, $\rho_\nu(z_1)$, $C_\nu(z_1)$, $\epsilon_\nu^{(3)}(z_1)$, $\sigma_\nu(z_1)$, and $\eta_{xyt}(z_1)$. Equations (28) give the values of two quantities at the end of the step, $\rho_\nu(z_1+h)$ and $C_\nu(z_1+h)$, and we can use these to calculate a third, $\epsilon_\nu^{(3)}(z_1+h)$. We have no way of finding the shape factors, $\sigma_\nu^{(3)}(z_1+h)$ and $\eta_{xyt}(z_1+h)$, but in practical problems these often remain nearly constant during propagation. In such cases we know all five values at the end of the step, and this allows us to reapply Eqs. (28) to determine the evolution of the beam for the second step. We repeat this process for the third step, the fourth step, and so forth to propagate through any arbitrarily large distance.

We can replace the chirp parameter in Eqs. (28) by using the radius of curvature, defined as $R_\nu = -z_{d\nu}/C_\nu$. The result is

$$\rho_\nu^2(z_1 + h) = \rho_{\nu1}^2\left[\left(1 + \frac{h}{R_{\nu1}}\right)^2 + \left(\frac{h}{z_{d\nu1}}\right)^2\left(\sigma_\nu^2 - \frac{\epsilon_\nu^{(3)}\eta_{xyt}}{d_\nu\beta_0}\,\frac{U}{P_c}\right)\right], \tag{29a}$$

$$\frac{1}{R_\nu(z_1 + h)} = \frac{\rho_{\nu1}^2}{\rho_\nu^2}\left[\frac{1}{R_{\nu1}}\left(1 + \frac{h}{R_{\nu1}}\right) + \frac{h}{z_{d\nu1}^2}\left(\sigma_\nu^2 - \frac{\epsilon_\nu^{(3)}\eta_{xyt}}{d_\nu\beta_0}\,\frac{U}{P_c}\right)\right]. \tag{29b}$$

The six coupled equations, Eqs. (28), or the six coupled equations, Eqs. (29), constitute the coupled SME's.

For a continuous-wave beam we can reduce the number of coupled equations from six to four. Similarly, if the pulse width does not change significantly during propagation, we can define an effective power as $P = U/\delta_t = U\eta_t/2\sqrt{\pi}\rho_t$, and again we can reduce the number of equations from six to four. In either case, for ν equal x or y the equations are

$$\rho_\nu^2(z_1 + h) = \rho_{\nu1}^2\left[\left(1 + \frac{h}{R_{\nu1}}\right)^2 + \left(\frac{h}{z_{d\nu1}}\right)^2\left(\sigma_\nu^2 - \epsilon_\nu^{(2)}\eta_{xy}\,\frac{P}{P_c}\right)\right], \tag{30a}$$

$$\frac{1}{R_\nu(z_1 + h)} = \frac{\rho_{\nu1}^2}{\rho_\nu^2}\left[\frac{1}{R_{\nu1}}\left(1 + \frac{h}{R_{\nu1}}\right) + \frac{h}{z_{d\nu1}^2}\left(\sigma_\nu^2 - \epsilon_\nu^{(2)}\eta_{xy}\,\frac{P}{P_c}\right)\right], \tag{30b}$$

where $\eta_{xy} = \eta_x\eta_y$ is the two-dimensional nonlinear shape factor and $\epsilon_\nu^{(2)} = \rho_\nu^2/\rho_x\rho_y$ is the two-dimensional ellipticity factor.

One can understand the physical meaning of the beam-quality factor σ_ν by considering Eq. (30a) for the case of linear propagation. From this equation the far-field angle of the beam initially at its beam waist is calculated to be $\theta = (\partial\rho_\nu/\partial z)_{z\to\infty} = \sigma_{\nu1}/2\beta_0\rho_{\nu1}$, which is just $\sigma_{\nu1}$ times the far-field angle of a Gaussian beam.

A special case of Eqs. (30) is for a circular beam, which has $\rho_x = \rho_y$ and $\epsilon_\nu^{(2)} = 1$. It has long been known that, for this case, Eqs. (30) are exact for any distance h and for any power P.[39,40] This fact is easily verified by use of procedures similar to those above to show that $\partial^m\rho_\nu^2/\partial z^m = 0$ for $m \geq 3$.

If the initial profile of a circular beam is Gaussian, then $\eta_{xy} = 1$ and $\sigma_x^2 = \sigma_y^2 = 1$, so that the rightmost parenthetical expression in Eqs. (30a) and (30b) goes to zero when $P = P_c$. Above this power a circular beam initially at a beam waist ($1/R_1 = 0$) will eventually shrink to an rms radius of zero. Similarly, if the initial radius of curvature is negative ($R_1 < 0$), which corresponds to a converging beam, then for $P = P_c$ the rms radius will shrink to zero when $z = |R_1|$. In reality, several complications accompany the catastrophic collapse that is due to self-focusing. One complication is that the central portion of the pulse undergoes catastrophic collapse before the rms radius goes to zero. For a circular Gaussian beam this occurs at $(3.77/4)P_c$.[41] Another complication is that the changes in the index of refraction depend on higher-order intensity-dependent effects ($\Delta n = n_2 I + n_4 I^2 + \cdots$).[42] Still another complication is that the paraxial (slowly varying amplitude) approximation breaks down when beam size becomes very small.[43,44]

Another special case is for the composite mean-squared width, defined as

$$\rho_{xy}^2 = \rho_x^2 + \rho_y^2. \tag{31}$$

To expand Eq. (31) we substitute Eq. (30a) for $\nu = x$ and $\nu = y$. The resulting formula is exact for any propagation distance h. This fact is easily verified by use of procedures similar to those above to show that $\partial^m\rho_{xy}^2/\partial z^m = 0$ for $m \geq 3$. The expanded form of Eq. (31) can be used to show that catastrophic collapse of an elliptical Gaussian beam occurs at $P = P_c(\epsilon_{x1} + \epsilon_{x1}^{-1})/2$, which is larger than the critical power P_c. The increased power of collapse for an elliptical beam has been demonstrated experimentally.[45]

The accuracy that can be obtained with second moments depends on the details of the problem. As we have seen, in some cases the SME's give exact results. In other cases the results are not exact, but they are close to those obtained with FFT's. This will be true whenever the shape factors associated with the beam do not change too much. This is often the case for self-mode-locked lasers, as we show in the accompanying paper.[26]

5. CONCLUSIONS

We have presented three computational techniques, two of which were derived for the first time to our knowledge, for including the effects of nonlinear coupling on the propagation of light through nonlinear, dispersive media. In the paper that follows[26] we compare the accuracy of these three techniques. We also show how to propagate through optical elements and inside cavities, and we demonstrate the importance of nonlinear coupling by considering a specific self-mode-locked laser as an example.

ACKNOWLEDGMENTS

This research was supported by the U.S. Army Research Office through a University Research Initiative Center and by IMRA America.

*Present address, SpatialMetriX Corporation, 222 Gale Lane, Kennett Square, Pennsylvania 19348.

REFERENCES AND NOTES

1. D. E. Spence, P. N. Kean, and W. Sibbett, “60-fsec pulse generation from a self-mode-locked Ti:sapphire laser,” Opt. Lett. **16,** 42–44 (1991).
2. J. Zhou, G. Taft, C.-P. Huang, M. M. Murnane, and H. C. Kapteyn, “Pulse evolution in a broad-bandwidth Ti:sapphire laser,” Opt. Lett. **19,** 1149–1151 (1994).

3. K. X. Liu, C. J. Flood, D. R. Walker, and H. M. van Driel, "Kerr lens mode locking of a diode-pumped Nd:YAG laser," Opt. Lett. **17,** 1361–1369 (1992).
4. G. P. A. Malcolm and A. I. Ferguson, "Self-mode locking of a diode-pumped Nd:YLF laser," Opt. Lett. **16,** 1967–1969 (1991).
5. N. H. Rizvi, P. M. W. French, and J. R. Taylor, "Generation of 33-fs pulses from a passively mode-locked Cr^{3+}: $LiSrAlF_6$ laser," Opt. Lett. **17,** 1605–1607 (1992).
6. A. Sennaroglu, C. R. Pollock, and H. Nathel, "Continuous-wave regeneratively mode-locked Cr:YAG laser," in *Conference on Lasers and Electro-Optics*, Vol. 8 of 1994 OSA Technical Digest Series (Optical Society of America, Washington, D.C., 1994), pp. 236–237.
7. F. Salin, J. Squier, and M. Piché, "Mode locking of $Ti{:}Al_2O_3$ lasers and self-focusing: a Gaussian approximation," Opt. Lett. **16,** 1674–1676 (1991).
8. D. Huang, M. Ulman, L. H. Acioli, H. A. Haus, and J. G. Fujimoto, "Self-focusing-induced saturable loss for laser mode locking," Opt. Lett. **17,** 511–513 (1992).
9. V. Magni, G. Cerullo, and S. De Silvestri, "*ABCD* matrix analysis of propagation of Gaussian beams through Kerr media," Opt. Commun. **96,** 348–355 (1993).
10. T. Brabec, P. F. Curley, Ch. Spielmann, E. Wintner, and A. J. Schmidt, "Hard-aperture Kerr-lens mode locking," J. Opt. Soc. Am. B **10,** 1029–1034 (1993).
11. F. Salin and J. Squier, "Gain guiding in solid-state lasers," Opt. Lett. **17,** 1352–1354 (1992).
12. M. Piché and F. Salin, "Self-mode locking of solid-state lasers without apertures," Opt. Lett. **18,** 1041–1043 (1993).
13. J. Herrmann, "Theory of Kerr-lens mode locking: role of self-focusing and radially varying gain," J. Opt. Soc. Am. B **11,** 498–512 (1994).
14. J. F. Cormier, M. Piché, and F. Salin, "Suppression of beam breakup in self-mode-locked Ti:sapphire lasers," Opt. Lett. **19,** 1225–1227 (1994).
15. H. W. Kogelnik, E. P. Ippen, A. Dienes, and C. V. Shank, "Astigmatically compensated cavities for cw dye lasers," IEEE J. Quantum Electron. **QE-8,** 373–379 (1972).
16. W. D. Johnston, Jr., and P. K. Runge, "An improved astigmatically compensated resonator for cw dye lasers," IEEE J. Quantum Electron. **QE-8,** 724–725 (1972).
17. R. E. Bridges, R. W. Boyd, and G. P. Agrawal, "Effect of beam ellipticity on self-mode locking in lasers," Opt. Lett. **18,** 2026–2028 (1993).
18. V. Magni, G. Cerullo, S. De Silvestri, and A. Monguzzi, "Astigmatism in Gaussian-beam self-focusing and in resonators for Kerr-lens mode locking," J. Opt. Soc. Am. B **12,** 476–485 (1995).
19. H. A. Haus, J. G. Fujimoto, and E. P. Ippen, "Structures for additive pulse mode locking," J. Opt. Soc. Am. B **8,** 2068–2076 (1991).
20. T. Brabec, Ch. Spielmann, and F. Krausz, "Mode locking in solitary lasers," Opt. Lett. **16,** 1961–1963 (1991).
21. I. P. Christov, M. M. Murnane, H. C. Kapteyn, J. Zhou, and C.-P. Huang, "Fourth-order dispersion-limited solitary pulses," Opt. Lett. **19,** 1465–1467 (1994).
22. J. L. A. Chilla and O. E. Martínez, "Spatial–temporal analysis of the self-mode-locked Ti:sapphire laser," J. Opt. Soc. Am. B **10,** 638–643 (1993).
23. B. Gross and J. T. Manassah, "Numerical solution for the propagation of an elliptic Gaussian beam in a Kerr medium," Phys. Lett. A **169,** 371–378 (1992).
24. A. M. Goncharenko, Yu. A. Logvin, A. M. Samson, and P. S. Shapovalov, "Rotating elliptical Gaussian beams in nonlinear media," Opt. Commun. **81,** 225–230 (1991).
25. F. Cornolti, M. Lucchesi, and B. Zambon, "Elliptic Gaussian beam self-focusing in nonlinear media," Opt. Commun. **75,** 129–135 (1990).
26. R. E. Bridges, R. W. Boyd, and G. P. Agrawal, "Multidimensional coupling owing to optical nonlinearities. II. Results," J. Opt. Soc. Am. B **13,** 560–569 (1996).
27. G. P. Agrawal, *Nonlinear Fiber Optics*, 2nd ed. (Academic, Boston, Mass., 1995), Chap. 2.
28. A. E. Siegman, *Lasers* (University Science Books, Mill Valley, Calif., 1986), Chaps. 7 and 8.
29. C. S. Liu, B. E. Cherrington, and J. T. Verdeyen, "Dispersion effects in a high-gain 3.39-μ He–Ne laser," J. Appl. Phys. **40,** 3556–3563 (1969).
30. R. W. Boyd, *Nonlinear Optics* (Academic, Boston, Mass., 1992), App. B.
31. S. A. Akmanov, R. V. Khokhlov, and A. P. Sukhorukov, "Self-focusing, self-defocusing, and self-modulation of laser beams," in *Laser Handbook*, F. T. Arrecchi and E. O. Schultz-DuBois, eds. (North-Holland, Amsterdam, 1992). What we call the critical power is the same as the "critical power" calculated in this reference by the method of moments.
32. J. H. Marburger, "Self-focusing: theory," Prog. Quantum Electron. **4,** 35–110 (1975). What we call the critical power is referred to in this reference as the power P_3 for a Gaussian beam.
33. Ref. 27, Chap. 4.
34. D. Marcuse, "Pulse distortion in single-mode fibers. 3: Chirped pulses," Appl. Opt. **20,** 3573–3579 (1981).
35. A.E. Siegman, "Output beam propagation and beam quality from a multimode stable-cavity laser," IEEE J. Quantum Electron. **29,** 1212–1217 (1993).
36. Ref. 28, Chap. 17.
37. M. A. Porras, J. Alda, and E. Bernabeu, "Complex beam parameter and *ABCD* law for non-Gaussian and non-spherical light beams," Appl. Opt. **30,** 6389–6402 (1992).
38. G. N. Lawrence, "Proposed international standard for laser-beam quality falls short," Laser Focus World **30**(7), 109–114, (1994).
39. C. Paré and P.-A. Bélanger, "Beam propagation in a linear or nonlinear lens-like medium using *ABCD* ray matrices: the method of moments," Opt. Quantum Electron. **24,** S1051–S1070 (1992). This reference uses the term "beam-quality factor" somewhat differently than we do. The difference has to do with the inclusion of the effect of the nonlinearity on the far-field divergence angle in the definition of beam-quality factor.
40. S. N. Vlasov, V. A. Petrishchev, and V. I. Talanov, "Averaged description of wave beams in linear and nonlinear media (the method of moments)," Radiophys. Quantum Electron. **14,** 1062–1070 (1971).
41. Ref. 32, Table 2.
42. V. M. Malkin, "On the analytical theory for stationary self-focusing of radiation," Physica D **64,** 251–266 (1993).
43. M. D. Feit and J. A. Fleck, Jr., "Beam nonparaxiality, filament formation, and beam breakup in the self-focusing of optical beams," J. Opt. Soc. Am. B **5,** 633–640 (1988).
44. N. Akhmediev, A. Ankiewicz, and J. M. Soto-Crespo, "Does the nonlinear Schrödinger equation correctly describe beam propagation?" Opt. Lett. **18,** 411–413 (1993).
45. C. R. Giuliano, J. H. Marburger, and A. Yariv, "Enhancement of self-focusing threshold in sapphire with elliptical beams," Appl. Phys. Lett. **21,** 58–60 (1972).