Supersensitive measurement of angular displacements using entangled photons

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We show that the use of path-entangled states of photons, having nonzero orbital angular momentum (OAM), increases the resolution and sensitivity of angular-displacement measurements performed using an interferometer. In the ideal case of maximally path-entangled states, the resolution of angular-displacement measurements increases by a factor of Nl, while the uncertainty in the measurement of angular displacements scales as $1/Nl$, where N is the number of entangled photons, half of which carry, on average, an OAM of $+l\hbar$ per photon and the other half carry an OAM of $-l\hbar$ per photon. We analyze measurement schemes for two- and four-photon entangled states produced by parametric down-conversion and, by employing a 4×4 matrix formalism to study the propagation of entangled OAM modes, obtain explicit expressions for the resolution and sensitivity in these schemes. These results constitute an improvement over what could be obtained with N nonentangled photons carrying an orbital angular momentum of $|l|\hbar$ per photon.

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I. INTRODUCTION

Precision measurements are important not only for verifying a given physical theory but also for possible applications of the theory. For example, the fact that relative displacements can be measured with subwavelength sensitivity through optical-phase measurements has led to many useful applications in a wide variety of fields including cosmology, nanotechnology, metrology, and medicine.

In generic classical schemes for optical-phase measurements, the sensitivity is limited by what is known as the standard quantum limit, which scales as $1/\sqrt{N}$, where N is either the average number of photons in the coherent-state input to the interferometer or the number of times the experiment is repeated with a one-photon Fock-state input [1,2]. More recent works have shown that the use of nonclassical states of light can lead to improved sensitivity in optical-phase measurements [3–6]. In particular, it has been shown that an N-photon entangled-state input to an interferometer gives rise to phase super-resolution [5,7–10], that is, the narrowing of interference fringes by N times compared to the fringes obtained with classical schemes at the same wavelength. It has also been shown that with N entangled photons the uncertainty in the estimation of optical phase scales as $1/N$, in contrast to the $1/\sqrt{N}$ scaling obtained using N nonentangled photons [11,12]. The $1/N$ scaling is known as the Heisenberg limit; beating the standard quantum limit by entangled photons is known as supersensitivity.

In this paper we consider an analogous type of measurement, namely, angular-displacement measurements. We seek to determine how accurately the angular orientation of an optical component can be measured using purely optical methods. Specifically, we consider an optical component in the form of a Dove prism and seek to measure its angular orientation by determining the rotation angle induced in an optical beam in passing through the prism. We assume that the prism is located in one arm of an interferometer. We thus seek to answer the question as to how accurately the angular displacements (rotations) introduced in a beam of light inside an interferometer can be measured. Measurements of this sort are generic to a broad class of problems in quantum metrology. We explicitly analyze measurement schemes for two- and four-photon entangled states produced by parametric down-conversion (PDC) and compare the angular resolution and sensitivity with those obtained using classical measurement schemes. We find that the use of entangled photons with nonzero orbital angular momentum increases the resolution and sensitivity of angular-displacement measurements.

The paper is organized as follows. In Sec. II we present a conceptual description of angular-displacement measurements with N independent single photons and also describe how the use of N entangled photons leads to increased resolution and sensitivity in angular-displacement measurement. In Sec. III we employ a 4×4 matrix formulation to study the propagation of entangled orbital angular momentum (OAM) modes through various optical elements and illustrate schemes for supersensitive angular-displacement measurements with two- and four-photon entangled states produced by PDC. Section IV presents our conclusions.

II. ANGULAR-DISPLACEMENT MEASUREMENTS: CONCEPTUAL DESCRIPTION

A. N independent single-photon states

Consider the situation shown in Fig. 1. N independent one-photon states with orbital angular momentum $l\hbar$ per photons go through a Mach-Zehnder interferometer. The two arms of the interferometer have within them two Dove prisms oriented at angles θ_1 and θ_2, respectively. A Dove prism is an optical element that rotates a beam carrying orbital angular momentum and also changes the sign of the beam's OAM mode index. Mathematically, a Dove prism introduces a phase of $\pi - 2l\theta$ in the path of the beam, where l is the OAM mode index of the beam and θ is the angle of rotation of the

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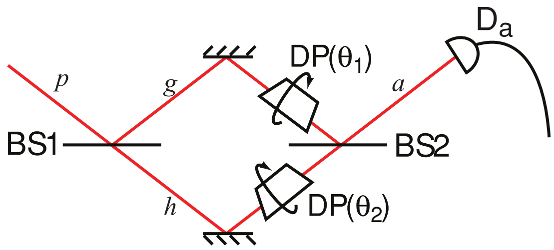

FIG. 1. (Color online) Scheme for angular-displacement measurement using N independent photons. The photons have an orbital angular momentum of $l\hbar$ per photon and are detected by the detector D_A in mode a. DP stands for Dove prism and BS denotes beam splitter.

prism [13]. We want to calculate the resolution and sensitivity with which the relative angular displacement $\theta = \theta_1 - \theta_2$ of the two Dove prisms can be measured.

Let us represent the state of the jth single photon in the input mode p by $|1_j\rangle_{p_{+l}}$. The subscript p_{+l} denotes the mode label p and also the OAM mode index $+l$. The state of the jth photon in the output mode a can be shown to be

$$|\phi_j\rangle = \tfrac{1}{2}(e^{-2il\theta_1} + e^{-2il\theta_2})|1_j\rangle_{a_{+l}}. \tag{1}$$

Since the photons are independent, we write the state $|\Psi\rangle$ of the N photons in the output port a of the interferometer as a direct product of the state of the N individual single photons, that is,

$$|\Psi\rangle = \prod_{j=1}^{N} |\phi_j\rangle. \tag{2}$$

Our measurement operator is the photon-number operator $\hat{N}_a$ in mode a,

$$\hat{N}_a = \sum_{k=1}^{N} \hat{a}^\dagger_{+l,k}\hat{a}_{+l,k}, \tag{3}$$

where $\hat{a}_{+l,k}$ is the creation operator corresponding to the kth photon in mode a. $\hat{N}_a$ detects the number of photons in mode a having an OAM of $l\hbar$ per photon. Following Ref. [14], we calculate the expectation value $\langle\hat{N}_a\rangle$ of the operator and the associated uncertainty $\langle\Delta\hat{N}_a\rangle^2$ to be

$$\langle\hat{N}_a\rangle = N\cos^2 l\theta, \tag{4}$$

$$\langle\Delta\hat{N}_a\rangle^2 = \frac{N}{4}\sin^2 l\theta, \tag{5}$$

where $\theta = \theta_1 - \theta_2$. The uncertainty $\Delta\theta_N^{(i)}$ in the measurement of θ is calculated as follows:

$$\Delta\theta_N^{(i)} = \frac{\langle\Delta\hat{N}_a\rangle}{|\partial\langle\hat{N}_a\rangle/\partial\theta|} = \frac{1}{2\sqrt{N}l}. \tag{6}$$

Here the superscript (i) indicates that the N single photons are independent. From Eq. (4) we find that the fringe spacing does not depend on N and so there is no enhancement of the resolution as a function of the number of independent photons. However, the sensitivity increases as the uncertainty $\Delta\theta_N^{(i)}$ does depend on N and scales as $1/\sqrt{N}$. This scaling is also known as the standard quantum limit.

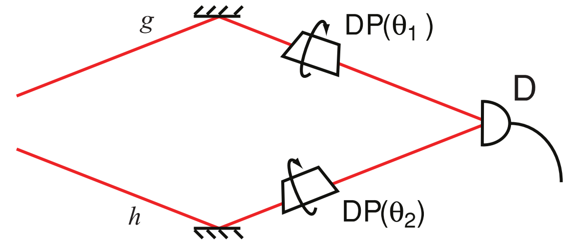

FIG. 2. (Color online) Angular-displacement measurement using N entangled photons. The photons in modes g and h are in the maximally entangled state and have an orbital angular momentum of $|l\hbar|$ per photon. The detector D is an N-photon detector.

B. *N* entangled photons

In this section we derive the angular resolution and sensitivity that can be obtained with N entangled photons in the ideal case (see Fig. 2). We consider the following entangled state of N photons in modes g and h of Fig. 2:

$$\begin{aligned}|\psi_N^l\rangle = \tfrac{1}{2}\big[&|N\rangle_{g_{+l}}|0\rangle_{h_{-l}} + |0\rangle_{g_{+l}}|N\rangle_{h_{-l}}\\ &+ |N\rangle_{g_{-l}}|0\rangle_{h_{+l}} + |0\rangle_{g_{-l}}|N\rangle_{h_{+l}}\big].\end{aligned} \tag{7}$$

Here $|N\rangle_{g_{+l}}|0\rangle_{h_{-l}}$ represents the quantum state with N photons in mode g, with orbital angular momentum $l\hbar$ per photon, and zero photons in mode h, etc. Such states are the generalizations of the $N00N$ states, which are the path-entangled states in a photon-number basis and which have been studied extensively in the context of supersensitive phase measurement [2,5,7,8]. In our scheme, the N-photon entangled state passes through a set of Dove prisms oriented at angles θ_1 and θ_2, respectively. The N-photon state $|\psi_N^l\rangle$ after the Dove prisms is therefore

$$\begin{aligned}|\psi_N^l\rangle = \tfrac{1}{2}[&e^{-2iNl\theta_1}|N\rangle_{g_{-l}}|0\rangle_{h_{-l}} + e^{2iNl\theta_2}|0\rangle_{g_{+l}}|N\rangle_{h+l}\\ &+ e^{2iNl\theta_1}|N\rangle_{g_{+l}}|0\rangle_{h_{+l}} + e^{-2iNl\theta_2}|0\rangle_{g_{-l}}|N\rangle_{h_{-l}}].\end{aligned} \tag{8}$$

We now calculate the angular resolution and sensitivity using the following measurement operator:

$$\begin{aligned}\hat{A_N} = &|N\rangle_{g_{+l}}|0\rangle_{h_{+l}\,g_{+l}}\langle 0|_{h_{+l}}\langle N|\\ &+ |0\rangle_{g_{+l}}|N\rangle_{h_{+l}\,h_{+l}}\langle N|_{h_{+l}}\langle 0|\\ &+ |N\rangle_{g_{-l}}|0\rangle_{h_{-l}\,g_{-l}}\langle 0|_{h_{-l}}\langle N|\\ &+ |0\rangle_{g_{-l}}|N\rangle_{h_{-l}\,g_{-l}}\langle N|_{h_{-l}}\langle 0|.\end{aligned} \tag{9}$$

This operator gives a direct measure of the degree of coherence between the N-photon states $|N\rangle_{g_{+l}}|0\rangle_{h_{-l}}$ and $|0\rangle_{g_{+l}}|N\rangle_{h_{-l}}$ and between the states $|N\rangle_{g_{-l}}|0\rangle_{h_{+l}}$ and $|0\rangle_{g_{-l}}|N\rangle_{h_{+l}}$. The expectation value of the above operator is

$$\langle\hat{A}_N\rangle = \cos 2Nl\theta, \tag{10}$$

where $\theta = \theta_1 - \theta_2$. By comparing Eqs. (4) and (10), we find that the resolution of the angular-displacement measurement is N times better than the resolution that can be obtained with N independent photons. Next, using the completeness relation $\langle\hat{A}_N^2\rangle = \mathbf{1}$, we calculate the uncertainty $\langle\Delta\hat{A}_N^2\rangle$ in the above measurement and find it to be $\langle\Delta\hat{A}_N^2\rangle = \sin^2 2Nl\theta$. The uncertainty $\Delta\theta$ in the angular-displacement measurement can now be shown to be

$$\Delta\theta_N = \frac{\langle \Delta \hat{A}_N \rangle}{|\partial\langle \hat{A}_N\rangle/\partial\theta|} = \frac{1}{2Nl}. \tag{11}$$

We find that the angular sensitivity increases with the number of entangled photons.

We note that to obtain the maximum resolution and sensitivity, the state of the photons in modes g and h has to have the generic form of Eq. (7). However, as we show in Sec. III, in most experimental schemes, the form of Eq. (7) can be obtained only for $N = 2$. For $N > 2$, the state in modes g and h always ends up with some additional, unwanted terms. Additional terms in the entangled state have detrimental effects on both the resolution and sensitivity. Although the detrimental effect on resolution can be overcome by choosing a suitable detection scheme, the effect on sensitivity cannot, in general, be overcome by the choice of the detection scheme.

III. SUPERSENSITIVE MEASUREMENT OF ANGULAR DISPLACEMENTS

In this section we describe in detail our measurement schemes for two and four entangled photons and derive the expressions for the resolution and sensitivity. Our measurement scheme, as depicted in Fig. 3, is based on the process of parametric down-conversion—a nonlinear optical process in which a pump photon at higher frequency breaks up into two entangled photons of lower frequencies. First, we derive the state of the OAM entangled photons produced by PDC.

A. Entangled photons produced by parametric down-conversion

We start with the following interaction Hamiltonian $\hat{H}(t)$ for PDC [15]:

$$\hat{H}(t) = \frac{\epsilon_0}{2}\int_{\mathcal{V}} d^3\mathbf{r}\,\chi^{(2)} E_0^{(+)}(\mathbf{r},t)\hat{E}_s^{(-)} \times (\mathbf{r},t)\hat{E}_i^{(-)}(\mathbf{r},t) + \text{H.c.}, \tag{12}$$

where $\mathcal{V}$ is the volume of the interacting part of the nonlinear crystal and $\chi^{(2)}$ is the second-order nonlinear susceptibility. $\hat{E}_j^{(+)}(\mathbf{r},t)$ and $\hat{E}_j^{(-)}(\mathbf{r},t)$ are the positive- and negative-frequency parts of the electric field, where $j = s$ and i stand for the signal and idler, respectively. The pump field E_0 is assumed to be strong and will therefore be treated classically. We decompose the three electric fields in terms of field modes

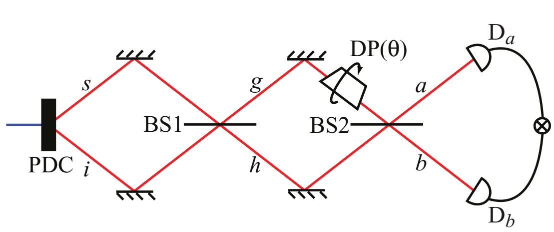

FIG. 3. (Color online) Scheme for supersensitive measurement of angular displacement using entangled photons. θ is the angle of rotation of the Dove prism in mode g. D_a and D_b are detectors set to detect photons with OAM mode indices $\pm l$ only.

$u_p^l(\mathbf{r})$ carrying an OAM. These modes are characterized by two indices l and p and carry an OAM of $l\hbar$ per photon owing to their azimuthal phase dependence of $e^{il\phi}$ [16]. The index l is referred to as the OAM mode index. The modes $u_p^l(\mathbf{r})$ are assumed to have the general form

$$u_p^l(\mathbf{r}) = R_p(\rho,z)\frac{e^{il\phi}}{\sqrt{2\pi}}, \tag{13}$$

with $R_p(\rho,z)$ being a complete set of orthonormal, radial modes, that is, $\sum_p \rho R_p(\rho,z)^* R_p(\rho',z) = \delta(\rho-\rho')$ and $\int \rho\, d\rho\, R_p(\rho,z)^* R_{p'}(\rho,z) = \delta_{pp'}$. Possible choices for $u_p^l(\mathbf{r})$ include the Laguerre-Gaussian modes, but the Schmidt modes of the down-converted field, which, in general, are not Laguerre-Gaussian modes [17], are usually the best choice. The three electric fields can now be written as

$$E_0^{(+)}(\mathbf{r},t) = A(\omega_0)u_{p_0}^{l_0}(\mathbf{r})e^{-i\omega_0 t}, \tag{14}$$

$$\hat{E}_s^{(-)}(\mathbf{r},t) = \sum_{l_s,p_s}\hat{s}_{l_s,p_s}^{\dagger}(\omega_s)u_{p_s}^{l_s *}(\mathbf{r})e^{i\omega_s t}, \tag{15}$$

$$\hat{E}_i^{(-)}(\mathbf{r},t) = \sum_{l_i,p_i}\hat{i}_{l_i,p_i}^{\dagger}(\omega_i)u_{p_i}^{l_i *}(\mathbf{r})e^{i\omega_i t}. \tag{16}$$

Here we have assumed that the signal, idler, and pump fields are monochromatic with frequencies ω_s, ω_i, and ω_0, respectively. $\hat{s}_{l_s,p_s}^{\dagger}$ denotes the creation operator corresponding to the signal mode, having photons with the OAM mode index equal to l_s and the radial mode index equal to p_s, etc. The three fields interact for some time within the nonlinear crystal and the state $|\psi\rangle$ of the down-converted photons after the interaction is given by $|\psi\rangle = \mathcal{T}\{\exp[(1/i\hbar)\int dt\,\hat{H}(t)]\}|\psi_0\rangle$, where $|\psi_0\rangle = |\text{vac}\rangle_s|\text{vac}\rangle_i$ is the initial vacuum state before the interaction, with no photons in either the signal or the idler mode. We assume perfect frequency phase matching such that $\omega_0 = \omega_s + \omega_i$; the symbol $\mathcal{T}$ represents operator time ordering. Taking the parametric interaction to be very weak, we then write the state $|\psi\rangle$ in terms of a perturbative expansion [15]:

$$|\psi\rangle = |\psi_0\rangle + |\psi_2\rangle + |\psi_4\rangle + \cdots. \tag{17}$$

The first term $|\psi_0\rangle$ is the initial vacuum state, the second term $|\psi_2\rangle \equiv \mathcal{T}[(1/i\hbar)\int dt\,\hat{H}(t)]|\psi_0\rangle$ is the two-photon state, the third term $|\psi_4\rangle \equiv \mathcal{T}[-(1/2\hbar^2)\int\int dt\,dt'\hat{H}(t)\hat{H}(t')]|\psi_0\rangle$ is the four-photon state, etc.

We calculate the two-photon state $|\psi_2\rangle$ by substituting Eqs. (12) and (14)–(16) into Eq. (17) and obtain

$$|\psi_2\rangle = \left(\frac{\epsilon_0\chi^{(2)}A(\omega_0)}{2i\hbar}\sum_{l_s,p_s}\sum_{l_i,p_i}\hat{s}_{l_s,p_s}^{\dagger}\hat{i}_{l_i,p_i}^{\dagger} \times \int_{\mathcal{V}} d^3\mathbf{r}\, u_{p_s}^{l_s *}(\mathbf{r})u_{p_i}^{l_i *}(\mathbf{r})u_{p_0}^{l_0}(\mathbf{r}) + \text{H.c.}\right)|\text{vac}\rangle_s|\text{vac}\rangle_i. \tag{18}$$

By working in the cylindrical coordinate system, using the orthogonality relation $\int_0^{2\pi} d\phi\, e^{i(l_0-l_s-l_i)\phi} = 2\pi\delta_{l_0,l_s+l_i}$, and taking $l_0 = 0$ for simplicity, we arrive at the following expression for the two-photon state:

$$|\psi_2\rangle = \sum_{l_s,p_s,p_i} \chi_{l_s,p_s,p_i} \hat{s}^{\dagger}_{l_s,p_s} \hat{i}^{\dagger}_{-l_s,p_i} |\text{vac}\rangle_s |\text{vac}\rangle_i, \tag{19}$$

where

$$\chi_{l_s,p_s,p_i} = \frac{\epsilon_0 \chi^{(2)} A(\omega_0)}{i2\sqrt{2\pi}\hbar} \times \iint \rho\, d\rho\, dz\, R_{p_s}(\rho,z) R_{p_i}(\rho,z) R_{p_0}(\rho,z) \tag{20}$$

is the probability amplitude that the signal and idler photons are in the modes characterized by indices l, p_s and $-l, p_i$, respectively. Next we consider a detection system that is insensitive to the radial indices and is sensitive only to the OAM mode index. In addition, we consider a class of states $|\psi_2^l\rangle$ that are obtained from Eq. (19) by keeping only terms with the OAM mode indices $\pm l$ for a given value of l. Thus we end up with the following normalized state of two photons:

$$\begin{aligned} |\psi_2^l\rangle &= \frac{1}{\sqrt{2}}(\hat{s}^{\dagger}_{+l}\hat{i}^{\dagger}_{-l} + \hat{s}^{\dagger}_{-l}\hat{i}^{\dagger}_{+l})|\text{vac}\rangle_s|\text{vac}\rangle_i \\ &= \frac{1}{\sqrt{2}}(|1\rangle_{s_{+l}}|1\rangle_{i_{-l}} + |1\rangle_{s_{-l}}|1\rangle_{i_{+l}}). \end{aligned} \tag{21}$$

Here we have explicitly separated a given mode into two different modes, one corresponding to the positive value of the orbital angular momentum and the other corresponding to the negative value. Thus $\hat{s}_{-l}$ represents the annihilation operator corresponding to the s mode having an OAM of $-l\hbar$, etc.

The next term in the expansion of Eq. (17) is the four-photon state $|\psi_4\rangle$. We evaluate this term in a similar manner and obtain

$$|\psi_4\rangle = \sum_{l_s,p_s,p_i} \sum_{l'_s,p'_s,p'_i} \chi_{l_s,p_s,p_i} \chi_{l'_s,p'_s,p'_i} \times \hat{s}^{\dagger}_{l_s,p_s} \hat{i}^{\dagger}_{-l_s,p_i} \hat{s}^{\dagger}_{l'_s,p'_s} \hat{i}^{\dagger}_{-l'_s,p'_i} |\text{vac}\rangle_s |\text{vac}\rangle_i. \tag{22}$$

Again assuming a detection system that is sensitive only to the OAM mode index and considering the class of states with the OAM mode indices $\pm l$, we obtain the normalized state

$$|\psi_4^l\rangle = \frac{1}{2\sqrt{3}}[\hat{s}^{\dagger}_{+l}\hat{s}^{\dagger}_{+l}\hat{i}^{\dagger}_{-l}\hat{i}^{\dagger}_{-l} + \hat{s}^{\dagger}_{-l}\hat{s}^{\dagger}_{-l}\hat{i}^{\dagger}_{+l}\hat{i}^{\dagger}_{+l} + 2\hat{s}^{\dagger}_{+l}\hat{s}^{\dagger}_{-l}\hat{i}^{\dagger}_{-l}\hat{i}^{\dagger}_{+l}]|\text{vac}\rangle_s|\text{vac}\rangle_i \tag{23}$$

or

$$|\psi_4^l\rangle = \frac{1}{\sqrt{3}}[|2\rangle_{s_{+l}}|2\rangle_{i_{-l}} + |2\rangle_{s_{-l}}|2\rangle_{i_{+l}} + |1\rangle_{s_{+l}}|1\rangle_{s_{-l}}|1\rangle_{i_{+l}}|1\rangle_{i_{-l}}]. \tag{24}$$

This is the four-photon entangled state.

B. Generic scheme for supersensitive angular-displacement measurement

In our measurement scheme, as outlined in Fig. 3, the entangled photons produced by PDC first get mixed at the beam splitter BS1. The photons in mode g then pass through a Dove prism oriented at angle θ and thereby get rotated with respect to the photons in mode h. The photons in modes g and h are then mixed at the second beam splitter BS2 and are subsequently detected in modes a and b by detectors D_a and D_b. Our aim is to determine the resolution and sensitivity with which the rotation angle θ can be measured.

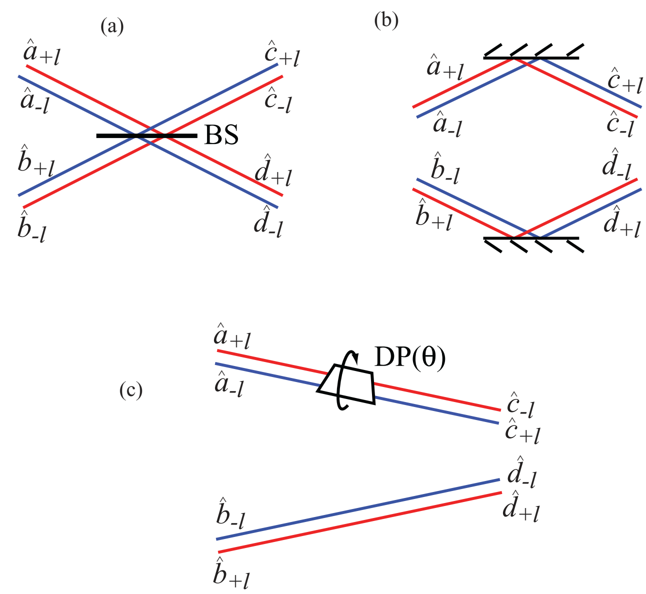

FIG. 4. (Color online) Transformation of OAM modes when they pass through (a) a beam splitter, (b) a pair of mirrors, and (c) a Dove prism.

We begin by summarizing the transformation properties of OAM modes when they pass through a beam splitter, a pair of mirrors, or a Dove prism. We note that, upon reflection, an OAM mode changes the sign of its mode index and also picks up an additional phase. This additional phase is equal to $\pi/2$ when the mode reflects from a symmetric beam splitter and π when it reflects from a mirror. Upon passage through a Dove prism, an OAM mode picks up an additional phase of $\pi - 2l\theta$ [13], where l is the orbital angular momentum mode index and θ is the angle of rotation of the Dove prism. The transformation matrices corresponding to the three optical elements can be calculated by employing a 4×4 matrix formulation. The beam-splitter transformation matrix is calculated in the following way. As shown in Fig. 4(a), let us suppose that a and b are the input modes to a beam splitter and c and d are the output modes. The annihilation operators corresponding to mode a are $\hat{a}_{+l}$ and $\hat{a}_{-l}$, etc. Using the standard beam-splitter operator algebra [18], we obtain the relationship between the input and output-mode annihilation operators and represent it as the matrix equation

$$\begin{pmatrix} \hat{c}_{+l} \\ \hat{c}_{-l} \\ \hat{d}_{+l} \\ \hat{d}_{-l} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \\ 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_{+l} \\ \hat{a}_{-l} \\ \hat{b}_{+l} \\ \hat{b}_{-l} \end{pmatrix} = M_{\text{BS}} \begin{pmatrix} \hat{a}_{+l} \\ \hat{a}_{-l} \\ \hat{b}_{+l} \\ \hat{b}_{-l} \end{pmatrix}. \tag{25}$$

Here the unitary matrix M_{BS} is the beam-splitter transformation matrix for OAM modes. In a similar manner, the transformation matrix M_{mir} related to the reflections of two incident modes a and b into the reflected modes c and d [Fig. 4(b)] can be shown to be

$$M_{\text{mir}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{26}$$

Finally, in situations in which one of the modes passes through a Dove prism [Fig. 4(c)], rotated at an angle θ, the transformation matrix M_{DP} is given by

$$M_{\text{DP}}(\theta) = \begin{pmatrix} 0 & -e^{2il\theta} & 0 & 0 \\ -e^{-2il\theta} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{27}$$

The first two diagonal elements are zero due to the fact that upon passage through a Dove prism a modes changes the sign of its OAM mode index. We note that both M_{mir} and M_{BS} are unitary matrices. Using the transformation properties of OAM modes as given by Eqs. (25)–(27), we express the output-mode annihilation operators in terms of the input-mode annihilation operators as

$$O = M_{\text{BS}} M_{\text{DP}}(\theta) M_{\text{mir}} M_{\text{BS}} M_{\text{mir}} I = M I, \tag{28}$$

where

$$O = \begin{pmatrix} \hat{a}_{+l} \\ \hat{a}_{-l} \\ \hat{b}_{+l} \\ \hat{b}_{-l} \end{pmatrix}, \quad I = \begin{pmatrix} \hat{s}_{+l} \\ \hat{s}_{-l} \\ \hat{i}_{+l} \\ \hat{i}_{-l} \end{pmatrix}.$$

In order to calculate the state in the output modes a and b, we need to obtain the inverse relationship, that is, we need to express the input-mode annihilation operators in terms of the output-mode annihilation operators. Therefore, we invert the above matrix equation and write it as

$$I = M^{-1} O = M^{\dagger} O, \tag{29}$$

where the last equality results from the fact that M is a unitary matrix ($M^{-1} = M^{\dagger}$), with $|\det M| = 1$, where det denotes the determinant. Now taking the transpose of Eq. (29), we obtain

$$I^{\dagger} = O^{\dagger} M. \tag{30}$$

We note that $I^{\dagger}$ and $O^{\dagger}$ are four-element row vectors: $I^{\dagger} = (\hat{s}^{\dagger}_{+l}, \hat{s}^{\dagger}_{-l}, \hat{i}^{\dagger}_{+l}, \hat{i}^{\dagger}_{-l})$ and $O^{\dagger} = (\hat{a}^{\dagger}_{+l}, \hat{a}^{\dagger}_{-l}, \hat{b}^{\dagger}_{+l}, \hat{b}^{\dagger}_{-l})$. Using Eqs. (25)–(27), we solve Eq. (30) to obtain the following operator relations:

$$\hat{s}^{\dagger}_{+l} = k_1^* \hat{a}^{\dagger}_{-l} + i k_2^* \hat{b}^{\dagger}_{+l}, \quad \hat{s}^{\dagger}_{-l} = k_1 \hat{a}^{\dagger}_{+l} + i k_2 \hat{b}^{\dagger}_{-l}; \tag{31a}$$

$$\hat{i}^{\dagger}_{+l} = i k_2 \hat{a}^{\dagger}_{+l} - k_1 \hat{b}^{\dagger}_{-l}, \quad \hat{i}^{\dagger}_{-l} = i k_2^* \hat{a}^{\dagger}_{-l} - k_1^* \hat{b}^{\dagger}_{+l}; \tag{31b}$$

where $k_1 = \frac{1}{2}(1 + e^{-2il\theta})$ and $k_2 = \frac{1}{2}(1 - e^{-2il\theta})$. With the operator relations in Eqs. (31), we next calculate the angular resolution and sensitivity that can be obtained with two and four entangled photons.

C. Supersensitive measurement with two entangled photons

The normalized two-photon state $|\psi_2^l\rangle$ produced by PDC is given by Eq. (19),

$$\begin{aligned} |\psi_2^l\rangle &= \sqrt{\tfrac{1}{2}}(|1\rangle_{s_{+l}}|1\rangle_{i_{-l}} + |1\rangle_{s_{-l}}|1\rangle_{i_{+l}}) \\ &= \sqrt{\tfrac{1}{2}}[\hat{s}^{\dagger}_{+l}\hat{i}^{\dagger}_{-l} + \hat{s}^{\dagger}_{-l}\hat{i}^{\dagger}_{+l}]|\text{vac}\rangle. \end{aligned} \tag{32}$$

Using the operator relations of Eqs. (31), we express the above state in terms of the output-mode creation operators to obtain

$$\begin{aligned} |\psi_2^l\rangle = \sqrt{\tfrac{1}{2}}[&(k_1^*\hat{a}^{\dagger}_{-l} + i k_2^* \hat{b}^{\dagger}_{+l})(i k_2^* \hat{a}^{\dagger}_{-l} - k_1^* \hat{b}^{\dagger}_{+l}) \\ &+ (k_1 \hat{a}^{\dagger}_{+l} + i k_2 \hat{b}^{\dagger}_{-l})(i k_2 \hat{a}^{\dagger}_{+l} - k_1 \hat{b}^{\dagger}_{-l})]|\text{vac}\rangle. \end{aligned} \tag{33}$$

We note that by carrying out a similar transformation between modes s and i and modes g and h, we can show that the state of the two photons in modes g and h is

$$\begin{aligned} |\psi_2^l\rangle = \frac{i}{2}[&|2\rangle_{g_{+l}}|0\rangle_{h_{-l}} + |0\rangle_{g_{+l}}|2\rangle_{h_{-l}} \\ &+ |2\rangle_{g_{-l}}|0\rangle_{h_{+l}} + |0\rangle_{g_{-l}}|2\rangle_{h_{+l}}]. \end{aligned} \tag{34}$$

This state has the same form as the state in Eq. (7) for $N = 2$; therefore, we expect this scheme to yield both maximum resolution and sensitivity.

We now estimate the angular resolution and sensitivity through use of the following measurement operator [9]:

$$\begin{aligned} \hat{A}_2 = &|1\rangle_{a_{+l}}|1\rangle_{b_{-l}\,a_{+l}}\langle 1|_{b_{-l}}\langle -1| \\ &+ |1\rangle_{a_{-l}}|1\rangle_{b_{+l}\,a_{-l}}\langle 1|_{b_{+l}}\langle 1|, \end{aligned} \tag{35}$$

which measures the probability of detecting either a photon in mode a with the OAM mode index l and another photon in mode b with the OAM mode index $-l$ or a photon in mode a with the OAM mode index $-l$ and another photon in mode b with the OAM mode index l. The measurement operator $\hat{A}_2$ does not experience the complete state $|\psi_2^l\rangle$; the effective state $|\psi_2^l\rangle_{\text{eff}}$ that $\hat{A}_2$ experiences is obtained from $|\psi_2^l\rangle$ by keeping only the terms containing $\hat{a}^{\dagger}_{+l}\hat{b}^{\dagger}_{-l}\hat{a}^{\dagger}_{-l}\hat{b}^{\dagger}_{+l}$. $|\psi_2^l\rangle_{\text{eff}}$ is given by

$$\begin{aligned} |\psi_2^l\rangle_{\text{eff}} &= -\frac{1}{\sqrt{2}}\left[\left(k_1^2 + k_2^2\right)\hat{a}_{+l}\hat{b}_{-l} + \left(k_1^{*2} + k_2^{*2}\right)\hat{a}_{-l}\hat{b}_{+l}\right]|\text{vac}\rangle \\ &= -\frac{1}{2\sqrt{2}}[(1 + e^{-4il\theta})|1\rangle_{a_{+l}}|1\rangle_{b_{-l}} \\ &\quad + (1 + e^{4il\theta})|1\rangle_{a_{-l}}|1\rangle_{b_{+l}}]. \end{aligned} \tag{36}$$

The expectation value of the measurement operator $\hat{A}_2$ is

$$\langle \hat{A}_2 \rangle = \text{Tr}\left[\hat{A}_2 |\psi_2^l\rangle\langle\psi_2^l|\right] = \cos^2(2l\theta). \tag{37}$$

We see that there is a twofold enhancement in the resolution of angular-displacement measurements. We note that, since there is no postselection involved in this case, the expectation value can be as high as unity for certain values of θ, indicating that at those values of θ the whole input state is detected at the output. Next, using $\langle \hat{A}_2^2 \rangle = \langle \hat{A}_2 \rangle = \cos^2(2l\theta)$, we obtain

$$\langle \Delta \hat{A}_2 \rangle = \sqrt{\langle \hat{A}_2^2 \rangle - \langle \hat{A}_2 \rangle^2} = \frac{\sin(4l\theta)}{2}. \tag{38}$$

Therefore, the uncertainty $\Delta\theta_2$ in the estimation of the angular displacement is

$$\Delta\theta_2 = \frac{\langle \Delta \hat{A}_2 \rangle}{|\partial \langle \hat{A}_2 \rangle / \partial \theta|} = \frac{1}{4l}. \quad (39)$$

By comparing the above result with Eq. (11), we see that this yields the maximum sensitivity that can be obtained with two entangled photons.

D. Supersensitive measurement with four entangled photons

We next consider the four-photon state $|\psi_4^l\rangle$ of Eq. (21),

$$|\psi_4^l\rangle = \frac{1}{2\sqrt{3}}[\hat{s}_{+l}^\dagger \hat{s}_{+l}^\dagger \hat{i}_{-l}^\dagger \hat{i}_{-l}^\dagger + \hat{s}_{-l}^\dagger \hat{s}_{-l}^\dagger \hat{i}_{+l}^\dagger \hat{i}_{+l}^\dagger + 2\hat{s}_{+l}^\dagger \hat{s}_{-l}^\dagger \hat{i}_{-l}^\dagger \hat{i}_{+l}^\dagger]|\text{vac}\rangle. \quad (40)$$

The propagated state in modes g and h can be shown to be

$$\begin{aligned}\left|\psi_4^l\right\rangle = -\frac{1}{\sqrt{24}}[&\sqrt{3}(|4\rangle_{g_{+l}}|0\rangle_{h_{-l}} + |0\rangle_{g_{+l}}|4\rangle_{h_{-l}} \\ &+ |4\rangle_{g_{-l}}|0\rangle_{h_{+l}} + |0\rangle_{g_{-l}}|4\rangle_{h_{+l}}) \\ &+ \sqrt{2}(|2\rangle_{g_{+l}}|2\rangle_{h_{-l}} + |2\rangle_{g_{-l}}|2\rangle_{h_{+l}} + |2\rangle_{g_{+l}}|2\rangle_{g_{-l}} \\ &+ |2\rangle_{g_{+l}}|2\rangle_{h_{+l}} + |2\rangle_{g_{-l}}|2\rangle_{h_{-l}} + |2\rangle_{h_{+l}}|2\rangle_{h_{-l}})]. \quad (41)\end{aligned}$$

We note that the above state does not have the same generic form of Eq. (7). This is due to the presence of the additional terms that are present in the second set of parentheses. We now quantify the effect of these additional term on the sensitivity and describe our detection scheme with the following four-photon measurement operator $\hat{A}_4$:

$$\hat{A}_4 = \hat{A}_4^{(I)} + \hat{A}_4^{(II)}, \quad (42)$$

where

$$\begin{aligned}\hat{A}_4^{(I)} = &|3\rangle_{a_{+l}}|1\rangle_{b_{-l}}\,{}_{a_{+l}}\langle 3|_{b_{-l}}\langle 1| \\ &+ |3\rangle_{a_{-l}}|1\rangle_{b_{+l}}\,{}_{a_{-l}}\langle 3|_{b_{+l}}\langle 1| \\ &+ |2\rangle_{a_{+l}}|1\rangle_{a_{-l}}|1\rangle_{b_{+l}}\,{}_{a_{+l}}\langle 2|_{a_{-l}}\langle 1|_{b_{+l}}\langle 1| \\ &+ |2\rangle_{a_{-l}}|1\rangle_{a_{+l}}|1\rangle_{b_{-l}}\,{}_{a_{-l}}\langle 2|_{a_{+l}}\langle 1|_{b_{-l}}\langle 1| \quad (43)\end{aligned}$$

and $\hat{A}_4^{(II)}$ is obtained by rewriting $\hat{A}_4^{(I)}$ with the indices a and b interchanged. The operator $\hat{A}_4$ measures the probability of detecting either three photons in mode a and one photon in mode b or one photon in mode a and three photons in mode b. The particular choice of the measurement operator is motivated by the fact that in order to achieve super-resolution, the four-photon measurement needs to postselect the ensemble that consists only of the maximally entangled four-photon states, that is, the state in Eq. (7) for $N = 4$.

The effective postselected state $|\psi_4^l\rangle_{\text{eff}}^{(I)}$ that the measurement operator $\hat{A}_4^{(I)}$ experiences is obtained by first expressing the state $|\psi_4^l\rangle$ in terms of the output-state creation operators using the operator relations given in Eqs. (31) and then keeping only the relevant terms. After a straightforward calculation we obtain

$$\begin{aligned}\left|\psi_4^l\right\rangle_{\text{eff}}^{(I)} = \frac{-i}{\sqrt{3}}[&k_1^* k_2^* (k_1^{*2} + k_2^{*2})\hat{a}_{+l}^\dagger \hat{a}_{+l}^\dagger \hat{a}_{+l}^\dagger \hat{b}_{-l}^\dagger \\ &+ k_1 k_2 (k_1^2 + k_2^2)\hat{a}_{-l}^\dagger \hat{a}_{-l}^\dagger \hat{a}_{-l}^\dagger \hat{b}_{+l}^\dagger \\ &+ k_1^* k_2^* (k_1^2 + k_2^2)\hat{a}_{-l}^\dagger \hat{a}_{-l}^\dagger \hat{a}_{+l}^\dagger \hat{b}_{-l}^\dagger \\ &\times k_1^* k_2^* (k_1^{*2} + k_2^{*2})\hat{a}_{+l}^\dagger \hat{a}_{+l}^\dagger \hat{a}_{-l}^\dagger \hat{b}_{+l}^\dagger]|\text{vac}\rangle. \quad (44)\end{aligned}$$

Substituting for k_1 and k_2, we get

$$\begin{aligned}\left|\psi_4^l\right\rangle_{\text{eff}}^{(I)} = \frac{-i}{8\sqrt{3}}[&\sqrt{6}(1 - e^{8il\theta})|3\rangle_{a_{-l}}|1\rangle_{b_{+l}} \\ &+ \sqrt{6}(1 - e^{-8il\theta})|3\rangle_{a_{+l}}|1\rangle_{b_{-l}} \\ &+ \sqrt{2}(e^{-4il\theta} - e^{4il\theta})|2\rangle_{a_{-l}}|1\rangle_{a_{+l}}|1\rangle_{b_{-l}} \\ &+ \sqrt{2}(e^{4il\theta} - e^{-4il\theta})|2\rangle_{a_{+l}}|1\rangle_{a_{-l}}|1\rangle_{b_{+l}}]. \quad (45)\end{aligned}$$

The expectation value $\langle \hat{A}_4^{(I)} \rangle$ of the measurement operator is calculated by substituting terms from Eqs. (43) and (45), which yields

$$\left\langle \hat{A}_4^{(I)} \right\rangle = \tfrac{1}{3}\sin^2(4l\theta). \quad (46)$$

By performing a similar calculation, we obtain $\langle \hat{A}_4^{(II)} \rangle = \frac{1}{3}\sin^2(4l\theta)$. Thus we get

$$\langle \hat{A}_4 \rangle = \left\langle \hat{A}_4^{(I)} \right\rangle + \left\langle \hat{A}_4^{(II)} \right\rangle = \tfrac{2}{3}\sin^2(4l\theta). \quad (47)$$

We note the fourfold enhancement in the angular resolution. However, we find that the maximum expectation value of the measurement operator in this case is only 2/3, which means that the inherent postselection of our detection scheme is throwing away 1/3 of the input photons. The uncertainty $\langle \Delta \hat{A}_4 \rangle$ is given by

$$\langle \Delta \hat{A}_4 \rangle = \sqrt{2/3}\sin(4l\theta)\sqrt{1 - (2/3)\sin^2(4l\theta)}. \quad (48)$$

The uncertainty $\Delta\theta_4$ in the estimation of angular displacement is therefore

$$\Delta\theta_4 = \frac{\langle \Delta \hat{A}_4 \rangle}{|\partial \langle \hat{A}_4 \rangle / \partial \theta|} = \frac{\sqrt{1 - (2/3)\sin^2(4l\theta)}}{8l\sqrt{2/3}\cos(4l\theta)}. \quad (49)$$

Figure 5 shows a plot of $\Delta\theta_4$ as a function of θ for $|l| = 1$. The dashed horizontal line is the minimum attainable uncertainty [Eq. (11)] with four entangled photons for $|l| = 1$. We note that, although the postselection has no effect on resolution,

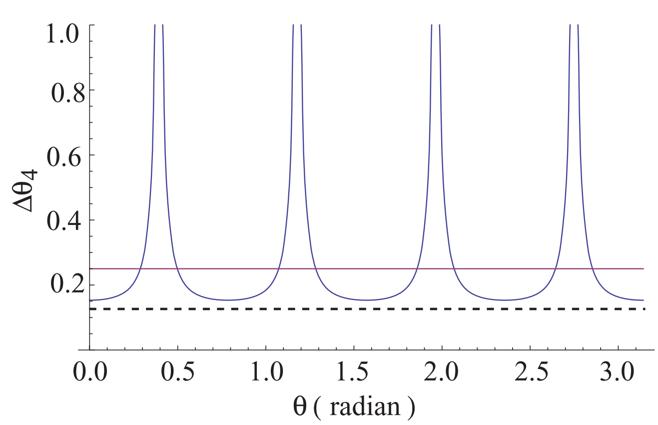

FIG. 5. (Color online) Plot of $\Delta\theta_4$ as a function of θ for $|l| = 1$. The solid horizontal line is the plot of $\Delta\theta_4^{(i)}$, as defined in Eq. (6), for four independent photons. The dashed horizontal line is the minimum attainable uncertainty with four entangled photons, as shown by Eq. (11).

it causes an increase in the minimum attainable uncertainly and thus a decrease in the maximum attainable sensitivity. The solid straight line is the uncertainly obtained with four independent photons for $|l| = 1$. We find that the uncertainty obtained with four entangled photons is lower than that obtained with four independent photons for a range of θ values. The lowest uncertainty, and thus the best sensitivity, is obtained near values of θ for which $4l\theta = n\pi$, where n is an integer.

IV. CONCLUSION

In conclusion, we have shown that the use of path-entangled states of photons, having nonzero orbital angular momentum, increases the resolution and sensitivity of angular-displacement measurements performed using an interferometer. We have found that the resolution of angular-displacement measurements increases by a factor Nl, while the uncertainty in the measurement of angular displacements scales as $1/Nl$, where N is the number of entangled photons and l is the magnitude of the orbital angular momentum mode index. Using a 4×4 matrix formulation to study the propagation of entangled OAM modes, we have explicitly analyzed measurement schemes for two and four entangled photons. It has previously been established [19–22] that the orbital angular momentum of light constitutes a useful degree of freedom for applications in quantum optics and quantum information science. The work presented here provides another such example. The ability to detect small rotations of optical components or of light beams themselves holds promise for many applications both in remote sensing and for performing fundamental studies of the propagation of light through optical materials.

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